1. Compute the radius and interval (including endpoints) of convergence for \( \sum_{n=1}^{\infty} \frac{(x+3)^n}{(5^n)(2n)} \).

\[
\lim_{n \to \infty} \left| \frac{\frac{(x+3)^{n+1}}{S^{n+1}(2n+2)}}{\frac{(x+3)^n}{S^n(2n)}} \right| = \lim_{n \to \infty} \left| \frac{(x+3)}{2n+2} \cdot \frac{2n}{S^n(2n)} \right| = \left| \frac{x+3}{5} \right|
\]

So, \( \left| \frac{x+3}{5} \right| < 1 \Rightarrow -1 < \frac{x+3}{5} < 1 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2 \).

Check endpoints:
- \( x = 2 \) gives \( \sum_{n=1}^{\infty} \frac{(2+3)^n}{5^n(2n)} = \sum_{n=1}^{\infty} \frac{1}{2n} \) (which diverges).
- \( x = -8 \) gives \( \sum_{n=1}^{\infty} \frac{(-8+3)^n}{5^n(2n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \), which passes the AST.

So, the interval is \([-8, 2]\) and radius is \(5\) (half the width of the interval).

2. Use a second-degree Taylor polynomial to estimate \( \sqrt{101} \).

\[
\begin{align*}
f(x) &= x^{1/2} \\
f'(x) &= \frac{1}{2}x^{-1/2} \\
f''(x) &= -\frac{1}{4}x^{-3/2} \\
f'''(x) &= \frac{3}{8}x^{-5/2} \\
f''''(x) &= -\frac{15}{16}x^{-7/2}
\end{align*}
\]

Choose \( a = 100 \) because we know \( \sqrt{100} = 10 \).

\[
\begin{align*}
\frac{f(100)}{2!} &= \frac{1}{20} \\
\frac{f''(100)}{4!} &= \frac{1}{6000} \\
P_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
&= 10 + \frac{1}{20}(x-100) + \frac{1}{6000}(x-100)^2 \\
P_2(101) &= 10 + \frac{1}{20} - \frac{1}{6000} \\
&= \frac{80399}{8000}
\end{align*}
\]

3. Find the complete Taylor series (in summation notation) for \( f(x) = \ln(1-x) \) about \( x = 0 \).

\[
\begin{align*}
f(x) &= \ln(1-x) \\
f'(x) &= -\frac{1}{1-x} \\
f''(x) &= -\frac{1}{(1-x)^2} \\
f'''(x) &= -\frac{2}{(1-x)^3} \\
f''''(x) &= -\frac{6}{(1-x)^4}
\end{align*}
\]

\[
\begin{align*}
f(0) &= 0 \\
f'(0) &= -1 \\
f''(0) &= -1 \\
f'''(0) &= -2 \\
f''''(0) &= -6
\end{align*}
\]

Series = \( (-1)^n x + \frac{-1}{2!} x^2 + \frac{-2}{3!} x^3 + \frac{-6}{4!} x^4 + \cdots \)

\[
= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots
\]

4. Evaluate the following exactly.

(a) \( 1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + \cdots \) = \( e^x \) series with \( x = -1 \) = \( e^{-1} \)

(b) \( 8/3 - 8/9 + 8/27 - 8/81 + \cdots \) = geometric: \( a = 8/3 \), \( r = -1/3 \) = \( \frac{a}{1-r} = \frac{8/3}{1-(-1/3)} \)

(c) \( \pi - \pi^3/6 + \pi^5/120 - \cdots \) = (sine series with \( x = \pi \)) = \( -\sin \pi = 0 \)
5. Let \( f(x) = x^3 \sin(-5x^2) \).

(a) Write out the first 3 non-zero terms in the Taylor series about \( x = 0 \) for \( f(x) \).

\[
f(x) = x^3 \left( \sin \omega \text{ with } \omega = -5x^2 \right)
\]
\[
= x^3 \left( -5x^2 - \frac{(-5x^2)^3}{3!} + \frac{(-5x^2)^5}{5!} - \ldots \right)
\]
\[
= x^3 \left( -5x^2 + \frac{125x^6}{6} - \frac{3125x^{10}}{120} + \ldots \right)
\]
\[
= -5x^5 + \frac{125x^9}{6} - \frac{3125x^{13}}{120} + \ldots
\]

(b) Write out the complete series for \( f(x) \) in summation notation.

\[
f(x) = \sum_{n=1}^{\infty} \frac{4n+1}{x} \cdot \frac{5^{2n-1}}{(2n-1)!} (-1)^n
\]

Or

\[
\sum_{n=0}^{\infty} \frac{4n+5}{x} \cdot \frac{5^{2n+1}}{(2n+1)!} (-1)^{n+1}
\]

or various other answers.

(c) Compute \( f^{(13)}(0) \).

After 13 derivatives, the first 2 terms have become 0, and all terms beyond the third term still contain \( x^4, x^8, x^{12} \), so they all become 0 when we plug in \( x = 0 \). Thus, only the third term remains, and after 13 derivatives, it will be \(-\frac{3125}{120}\).

(d) Compute \( \lim_{x \to 0} \frac{5x^5 + f(x)}{5x^9} \).

\[
= \lim_{x \to 0} \frac{5x^5 - 5x^5 + \frac{125x^9}{6} - \frac{3125x^{13}}{120}}{5x^9}
\]

\[
= \lim_{x \to 0} \left( \frac{25}{6} - \frac{625x^4}{120} \right)
\]

\[
= \frac{25}{6}
\]

6. Use the appropriate second-degree Taylor polynomials to estimate a solution near \( x = 0 \) to \( 1 + \sin 8x = e^{10x} \). Since \( \sin 8x \approx 8x \) and \( e^{10x} \approx 1 + 10x + \frac{(10x)^2}{2} = 1 + 10x + 50x^2 \),

\[
8x = \sqrt{1 + 10x + 50x^2}
\]

\[
0 = 2x + 50x^2
\]

\[
0 = 2x(1 + 25x) \implies x = 0, x = -\frac{1}{25}
\]
7. Find the general solution of the differential equation \( \frac{dy}{dx} (1 + x^3) = x^2 e^{7y} \).

\[
\int e^{-7y} dy = \int \frac{1}{w} dw = \frac{1}{3} \ln |w| + C \\
\int \frac{e^{-7y}}{1 + x^3} dx = \int \frac{3}{w} dw = x^2 + D \\
-7y = \ln \left( \frac{-7}{3} \ln (1 + x^3) + D \right)
\]

How could you check that your solution is correct? Plug \( y \) and its derivative into the original D.E. and make sure both sides of the D.E. are equal.

8. Sketch the slope field for \( \frac{dy}{dx} = y - 4x \).

9. Use Euler’s Method with 3 steps to estimate \( y(3/4) \) if \( \frac{dy}{dx} = y - 4x \) and \( y(0) = 2 \) and decide if your answer is too large or too small.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \frac{dy}{dx} )</th>
<th>( \Delta x \cdot \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2 \cdot \frac{1}{4} = \frac{1}{2}</td>
<td>\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}</td>
</tr>
<tr>
<td>\frac{1}{4}</td>
<td>\frac{5}{2}</td>
<td>\frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}</td>
<td>\frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}</td>
</tr>
<tr>
<td>\frac{1}{2}</td>
<td>\frac{23}{8}</td>
<td>\frac{7}{8} \cdot \frac{1}{4} = \frac{7}{32}</td>
<td>\frac{7}{32} \cdot \frac{1}{4} = \frac{7}{128}</td>
</tr>
<tr>
<td>\frac{3}{4}</td>
<td>\frac{99}{32}</td>
<td>\frac{11}{4} \cdot \frac{1}{4} = \frac{11}{16}</td>
<td>\frac{11}{16} \cdot \frac{1}{4} = \frac{11}{64}</td>
</tr>
</tbody>
</table>

So, \( y \left( \frac{3}{4} \right) \approx \frac{99}{32} \).

Since curve is concave down (see #8), this is too large.
10. A colony of endangered sea otters has an annual birth rate of 10% and an annual death rate of 15%. In an attempt to sustain the colony, activists bring in otters from another region where the animals are plentiful. They do this at a rate of 50 otters per year.

(a) Write a DE whose solution is \( P(t) \), the otter population \( t \) years from now.

\[
\frac{dP}{dt} = .1P - .15P + 50 = -.05P + 50 \rightarrow \frac{dP}{dt} = -.05P + 50
\]

(b) Find any and all equilibrium solutions.

\[ 0 = -.05(P - 1000) \Rightarrow P = 1000 \]

birth rate = .1(100/yr)

death rate = -.15(20/yr)

imp. rate = +50/yr

(c) Find the general solution of your DE.

\[
\int \frac{dP}{P - 1000} = \int -.05 \, dt
\]

\[
\ln|P - 1000| = -.05t + C
\]

\[
|P - 1000| = e^{-.05t} e^C
\]

\[
P - 1000 = \pm e^C e^{-.05t}
\]

\[ 400 = 1000 + Ae^0 \]

\[ -600 = A \Rightarrow P = 1000 - 600e^{-0.05t} \]

(d) Find and sketch the particular solution if the current population is 400 otters.

![Graph showing stable equilibrium]

11. A population obeys the differential equation \( \frac{dP}{dt} = .001P(3000 - P) \). Sketch solutions for \( P(t) \) for the following initial populations: \( P(0) = 0 \), \( P(0) = 100 \), \( P(0) = 2000 \), \( P(0) = 4000 \).