Final Exam, Math 205A (Linear Algebra)

This take-home exam is due by 5 PM on **Friday, December 13**. You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the appropriate place on the other side of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Matrix multiplications and reduced row echelon forms may be done on MATLAB or a calculator, but please show all other work.

1. (14 points) Find the complete solution of the system $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & 1 & 5 \\ 5 & 10 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -7 \end{pmatrix}$.

2. (18 points) Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 6 & 12 & 14 \\ 4 & 21 & 21 \end{pmatrix}$.

(a) Find the LU factorization of A.

(b) Find the determinant of L and the determinant of U.

(c) Use your answer to (b) to find the determinant of A.

(d) Use your answer to (a) to solve $A\vec{x} = \begin{pmatrix} 13 \\ 40 \\ 37 \end{pmatrix}$.

3. (18 points) Find a basis for each of the four subspaces associated with the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 3 & 5 & 3 & 1 \end{pmatrix}$$

What is the factored form of A that displays these bases?

4. (18 points) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} 5 \\ 4 \\ 1 \\ -2 \end{pmatrix}$, and let S be the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$. Find the matrix P that projects vectors in $\mathbb{R}^4$ onto S, and the matrix R that reflects vectors in $\mathbb{R}^4$ through S. Also find the projection $\vec{p}$ of $\vec{v}_3$ onto S, and the reflection $\vec{r}$ of $\vec{v}_3$ through S.

5. (16 points) As in project #2, let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be a unit vector in $\mathbb{R}^3$ and define $U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$.

(i) How is U^T related to U? Matrices with this property are called **skew symmetric**.

(ii) What is $U\vec{u}$? What is $\vec{u}^T U$?

(iii) Explain how you can tell that $\vec{u}\vec{u}^T$ is a projection matrix. What does it project onto?

(iv) Show that $U^2 = \vec{u}\vec{u}^T - I$.

6. (12 points) As in project #2, the matrix

$$R = (\cos\theta)\, I + (1 - \cos\theta)\, \vec{u}\vec{u}^T + (\sin\theta)\, U$$

rotates vectors in $\mathbb{R}^3$ counterclockwise by θ about the line through $(0, 0, 0)$ in the direction of $\vec{u}$, where $\vec{u}$ and U are as in problem 5. Show that R is an orthogonal matrix, *i.e.*, $R^{-1} = R^T$. I can see two logical ways to go about this: either try to figure out what R^{-1} must be, and show that R^T is the same thing (some parts of problem 5 may be helpful here); or calculate $R^T R$ with the aid of problem 5 and see that you get I. If you choose the first way (which may be a lot less work), please explain your answer carefully.

7. (14 points) Let $A = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix}$.

(i) Calculate the eigenvalues and the corresponding eigenvectors of A.

(ii) For each eigenvalue λ, find a basis for the column space of $A - \lambda I$. What do you notice?

8. (20 points) Let $R = \frac{1}{5}\begin{pmatrix} -4 & 2 & 1 & 2 \\ 2 & 4 & 2 & -1 \\ 1 & 2 & -4 & 2 \\ 2 & -1 & 2 & 4 \end{pmatrix}$.

(i) Explain how you can tell that R is a reflection matrix.

(ii) Let S be the subspace of $\mathbb{R}^4$ that R reflects through. If $\vec{v}$ is in S, what should $R\vec{v}$ be? Explain.

(iii) The answer to (ii) implies that we can find a basis for S by finding the nullspace of $R - I$. Explain why.

(iv) Find a basis for S using the idea in (iii).

(v) In doing (iv) you must also have found a basis for the row space of $R - I$. What significance does it have?

(vi) Find the eigenvalues and eigenvectors of R. Hint: this should require little or no extra calculation.

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________