Math 205 A B Fall 2012 NAME (legibly!) Suggested solutions

FINAL EXAM December 12, 2012 Circle Your SECTION: A (8 am) B (9:30 am)

DO NOT WRITE HERE!

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TOTAL	

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!

1. Let **F** be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that we have been using in class.
Let $H = \{f \in \mathbf{F} \mid f$ has the same y-coordinate at $x = -1$ as it does at $x = 1\}$, so a vector **u** belongs to H iff **u** is a function f satisfying $f(-1) = f(1)$.

(1A) Give an example of a *non-constant* function f belonging to H.

For example: let $f(x) = x^2+1$. Then $f(-1) = (-1)^2+1 = 1+1 = 2$
and $f(1) = 1^2+1 = 1+1 = 2$
So $f(-1) = f(1)$

(1B) It's true that H a subspace of **F**. For just the first TWO of the three conditions given in the three parts of the definition of a subspace (in the order we've always talked about them), prove that H satisfies that condition.

1Bi) proof that the first condition, or part, of the subspace definition holds:

That $\vec{0} \in H$. Here $\vec{0}$ is the constant function $g(x) = 0$ for all $x \in \mathbb{R}$.
In particular, $g(-1) = 0$ and $g(1) = 0$ so $g(-1) = g(1)$.
Thus g, or the $\vec{0}$, belongs to H.

1Bii) proof that the second condition of the subspace definition holds:

That $\vec{u}$ and $\vec{v}$ each belong to $H \Longrightarrow \vec{u} + \vec{v} \in H$.

So: suppose $\vec{u}$ and $\vec{v}$ are in H.

Then $\vec{u}$ is a fn f satisfying $f(-1) = f(1)$
and $\vec{v}$ is a fn g satisfying $g(-1) = g(1)$.
Now $\vec{u}+\vec{v}$ is the function K defined by $K(x) = f(x)+g(x)$.

$$\begin{aligned} \text{So } K(-1) &= f(-1) + g(-1) \\ &= f(1) + g(1) \\ &= K(1) \qquad (\text{so, } K(-1) = K(1)) \end{aligned}$$

so K, or, $\vec{u}+\vec{v}$, is in H.

(the third part, $\vec{u} \in H$ and $\alpha \in \mathbb{R} \Rightarrow \alpha\vec{u} \in H$, you didn't have to show)

2. Let $\mathcal{S}$ be the vector space of all sequences $\mathbf{s} = (s_1,\ s_2,\ s_3,\ \ldots)$ of real numbers that we've discussed in class, and $\mathbf{F}$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define $T : \mathbf{F} \to \mathcal{S}$ by

$$T(f) = (f(1) + 2,\quad f(2) + 3,\quad f(3) + 4,\quad f(4) + 5,\ \ldots).$$

For example, if $f(x) = x^2$, then the *second* term in sequence $T(f)$ is $f(2) + 3 = 2^2 + 3 = 4 + 3 = 7$. The first two terms of $T(f)$ are then $(3,\ 7,\ \ldots)$.

(2A) So, if $f(x) = x^2$, what are the first *five* terms of the sequence $T(f)$?

$$T(f) = (f(1)+2,\ f(2)+3,\ f(3)+4,\ f(4)+5,\ f(5)+6, \cdots) = (1^2+2,\ 2^2+3,\ 3^2+4,\ 4^2+5,\ 5^2+6, \cdots)$$
$$= (1+2,\ 4+3,\ 9+4,\ 16+5,\ 25+6, \cdots)$$
$$= (3,\ 7,\ 13,\ 21,\ 31, \cdots)$$

(2B) Find $T(g)$ where $g(x) = x^3$. (Give the first five terms).

$$= (1^3+2,\ 2^3+3,\ 3^3+4,\ 4^3+5,\ 5^3+6, \cdots) = (1+2,\ 8+3,\ 27+4,\ 64+5,\ 125+6 \cdots)$$
$$= (3,\ 11,\ 31,\ 69,\ 131, \cdots)$$

(2C) Find $T(5f) = T(5x^2)$ through the first five terms.

$$= (5\cdot 1^2+2,\ 5\cdot 2^2+3,\ 5\cdot 3^2+4,\ 5\cdot 4^2+5,\ 5\cdot 5^2+6, \cdots) = (7,\ 23,\ 49,\ 85,\ 131, \cdots)$$

(2D) Is T a linear transformation? For each of the two parts of the definition of a linear transformation, either prove that T satisfies that part, or show it does not by giving a specific counterexample. I'd recommend considering the examples you've worked on in 2A–2C!

(2Di) (part one of the LT definition: your proof or counterexample):

let's see if $T(x^2+x^3) = T(x^2)+T(x^3)$.

now, $T(x^2+x^3) = (1^2+1^3+2,\ 2^2+2^3+3,\ 3^2+3^3+4,\ 4^2+4^3+5,\ 5^2+5^3+6, \cdots$

$= (1+1+2,\ 4+8+3,\ 9+27+4,\ 16+64+5,\ 25+125+6, \cdots$

$= (4,\ 15,\ 40,\ 85,\ 156, \cdots)$

where as $T(x^2) + T(x^3) = (3,\ 7,\ 13,\ 21,\ 31, \cdots)$

$+ (3,\ 11,\ 31,\ 69,\ 131, \cdots)$

$= (6,\ 18,\ 44,\ 90,\ 162, \cdots)$

which is NOT equal to $T(x^2+x^3)$ so NO T is not a L.T.

(2Dii) (part two of the LT definition: your proof or counterexample):

we suspect failure here as well: let's see if $T(5x^2) = 5\,T(x^2)$

now, $T(5x^2) = (7,\ 23,\ 49,\ 85,\ 131, \cdots)$

and $5\,T(x^2) = 5(3,\ 7,\ 13,\ 21,\ 31 \cdots)$

$= (15,\ 35,\ 65,\ 105,\ 155 \cdots)$

so $T(5x^2) \neq 5\,T(x^2)$; $\therefore T$ fails the 2nd part of the defn of L.T. as well as the 1st.

3. Let $A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 2 & 4 \\ 1 & -3 & 1 \end{bmatrix}$; let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the column vectors of A, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(3A). Explain why $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for any $\mathbf{b} \in \mathbb{R}^3$.

rref of $[A|\vec{b}]$ has the form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \vec{b}^* \\ 0 & 0 & 1 & \end{array}\right]$. But no matter what $\vec{b}^*$ is, the system represented by $[A|\vec{b}]$ is consistent ("no rows of 0's") so $A\vec{x} = \vec{b}$ has a soln.

Alternatively: since rref$(A) = I_3$, A^{-1} exists so $\vec{x} = A^{-1}\vec{b}$; ie. $A\vec{x} = \vec{b}$ has a soln' (THE soln' $\vec{x} = A^{-1}\vec{b}$)

(3B). Explain why the set S is linearly independent.

again, rref$(A) = I_3$ shows the only soln' to $A\vec{x} = \vec{0}$ is $x_1 = x_2 = x_3 = 0$, ie the only soln' to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ is $x_1 = x_2 = x_3 = 0$.

(3C). Parts 3A and 3B say that S is a basis for $\mathbb{R}^3$. Show that it is an orthogonal basis.

you need to show the column vectors are pairwise $\perp$.
you can check $\vec{v}_1 \cdot \vec{v}_2 = 0$, etc. by hand OR understand why its enough to see that A^TA has the form $\begin{bmatrix} * & 0 & 0 \\ & * & 0 \\ & & * \end{bmatrix}$!

(3D.) Since S is an orthogonal basis of $\mathbb{R}^3$, there's a formula that uses dot-products to find the weights needed to express a vector $\mathbf{u} \in \mathbb{R}^3$ as a LC of the members of S. What is that formula?

$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{u}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3$$

(3E). Use the formula in 3D to express $\mathbf{u} = \begin{bmatrix} -333 \\ 493 \\ -109 \end{bmatrix}$ as a linear combination of the vectors in S. Show all your work. Express all weights as fractions in lowest terms.

Let $B = [A|\vec{u}]$; then $B^TB = \begin{bmatrix} 3 & 0 & 0 & 51 \\ 0 & 14 & 0 & 980 \\ 0 & 0 & 42 & 3528 \\ 51 & 980 & 3528 & 365819 \end{bmatrix}$

has all the dot products we need (and some extras ... what IS this?)

$$\text{so } \vec{u} = \frac{51}{3}\vec{v}_1 + \frac{980}{14}\vec{v}_2 + \frac{3528}{42}\vec{v}_3$$

$$= 17\vec{v}_1 + 70\vec{v}_2 + 84\vec{v}_3$$

[an easy way to check this is find rref$([A|\vec{u}])$]

4. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$. Let $\mathbf{y} = \begin{bmatrix} 9 \\ 18 \\ 28 \\ 4 \end{bmatrix}$.

(4A) It turns out that $\mathbf{y}$ is not in the column space of A, but you do not need to check this. Find the least squares solution of $A\mathbf{x} = \mathbf{y}$. Show all your work.

the least squares solution (LSQ) is obtained by solving $A^T A \vec{x} = A^T \vec{y}$

the corresponding aug. matrix is $\left[\begin{array}{ccc|c} 54 & 23 & 39 & 194 \\ 23 & 10 & 16 & 81 \\ 39 & 16 & 30 & 145 \end{array}\right]$

its rref is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$

which represents $\vec{X} = \begin{bmatrix} 7 \\ -8 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ where x_3 is free

(4B) Find the projection of $\mathbf{y}$ onto the column space of A. Note well the columns are *not* orthogonal!

This is $\vec{w} = A\vec{x}$ for any $\vec{x}$ above (infinitely many $\vec{x}$'s but all lead to the same $\vec{w}$)

so $\vec{w} = A \begin{bmatrix} 7 \\ -8 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 13 \\ 19 \end{bmatrix}$

(4B) Find the distance from $\mathbf{y}$ to $\mathrm{Col}(A)$. Show all your work.

this is $\|\vec{z}\|$ where $\vec{z} = \vec{y} - \vec{w}$ $\quad (\vec{z} \in \mathrm{Col}(A)^{\perp})$

$$= \begin{bmatrix} 9 \\ 18 \\ 28 \\ 4 \end{bmatrix} - \begin{bmatrix} 6 \\ 12 \\ 13 \\ 19 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 15 \\ -15 \end{bmatrix} = \vec{z}$$

so $\|\vec{z}\| = \sqrt{3^2 + 6^2 + 15^2 + (-15)^2}$

$= \sqrt{495} \approx 22.2485\ldots$

$(= 3\sqrt{55})$

5. In this problem, in your answers and work, write all repeating decimal numbers to **4 places** after the decimal point (not fractions). For example, write 1/3 as 0.3333…. But maintain complete precision in the calculator itself.

There is no parabola passing through the points (2, 3) (3, 5) (4, $3\frac{1}{6}$) (6, $6\frac{1}{3}$); you do *not* need to verify this. [NOTE that $3\frac{1}{6}$ means $19/6 = 3.1666666\ldots$ and $6\frac{1}{3} = 19/3 = 6.333333\ldots$ — enter such numbers into your calculator as fractions (eg, 19/6 and 19/3) to maintain maximum precision].

(5A) Find the parabola $y = \beta_2 x^2 + \beta_1 x + \beta_0$ that is the "best-fit" parabola for these points. Show all your work, including all matrices and vectors involved in solving this problem. Remember: write just **4 places** for repeating decimal expansions.

we're TOLD then that $\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}}_{\vec{\beta}} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 3\frac{1}{6} \\ 6\frac{1}{3} \end{bmatrix}}_{\vec{y}}$ has no sol'n (recall this matrix eq'n results from the 4 eqns obtained if each data point is "plugged into" $\beta_2 x^2 + \beta_1 x + \beta_0 = y$)

(just as a check, rref of $[X \mid \vec{y}]$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ showing the matrix eq'n $X\vec{\beta} = \vec{y}$ is inconsistent, ie. there is no parabola containing those four pts)

we must be content with the least squares sol'n:

solve $X^T X \vec{\beta} = X^T \vec{y}$. The corresponding aug. matrix:

$$\left[\begin{array}{ccc|c} 1649 & 315 & 65 & 335.666\bar{6} \\ 315 & 65 & 15 & 71.666\bar{6} \\ 65 & 15 & 4 & 17.5 \end{array}\right]$$

its rref: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & .166\bar{6} \\ 0 & 1 & 0 & -.666\bar{6} \\ 0 & 0 & 1 & 4.166\bar{6} \end{array}\right] \Rightarrow$ the best fit parabola is

$y = .166\bar{6}x^2 - 0.666\bar{6}x + 4.166\bar{6}$

OR

$\left(y = \frac{1}{6}x^2 - \frac{2}{3}x + 4\frac{1}{6}\right)$

(5B) What are the predicted values, that is, the y-coordinates of this best-fit parabola at $x = 2$, 3, 4 and 6, respectively?

we understand these are given by $X \begin{bmatrix} .166\bar{6} \\ -.666\bar{6} \\ 4.166\bar{6} \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.666\bar{6} \\ 4.166\bar{6} \\ 6.166\bar{6} \end{bmatrix} \quad \left(= \begin{bmatrix} 3\frac{1}{2} \\ 3\frac{2}{3} \\ 4\frac{1}{6} \\ 6\frac{1}{6} \end{bmatrix} \right)$

(the projection of the vector $\vec{y}$ onto col(X))

(5C) What is the sum-of-the-squares ("SOS") of the residuals for the predicted values found in (5B)? Show all your computations.

that's $\left\| \begin{pmatrix} \text{observed values} \\ \text{vector} \end{pmatrix} - \begin{pmatrix} \text{predicted values} \\ \text{vector} \end{pmatrix} \right\|^2 = \left\| \begin{bmatrix} 3 \\ 5 \\ 3\frac{1}{6} \\ 6\frac{1}{3} \end{bmatrix} - \begin{bmatrix} 3\frac{1}{2} \\ 3\frac{2}{3} \\ 4\frac{1}{6} \\ 6\frac{1}{6} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -0.5 \\ 1.333\bar{3} \\ -1.0 \\ 0.166\bar{6} \end{bmatrix} \right\|^2 =$ (next line)

$= (-0.5)^2 + \cdots + (0.166\bar{6})^2 = 3.05\bar{5}$

(5D) A polynomial whose coefficients are "sort of close" to ~~the best-fit~~ coefficients is $y = 0.2\cancel{x}\,x^2 - 0.5x + 4$. If this polynomial were used to find the predicted values, what would they be?

these are the 4 entries, respectively, of $X \begin{bmatrix} 0.2 \\ -0.5 \\ 4.0 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 4.3 \\ 5.2 \\ 8.2 \end{bmatrix}$

(5E) What is the SOS of the residuals for the predicted values in 5D?

it's again the dot product of $\left(\begin{bmatrix} 3 \\ 5 \\ 3\frac{1}{6} \\ 6\frac{1}{3} \end{bmatrix} - \begin{bmatrix} 3.8 \\ 4.3 \\ 5.2 \\ 8.2 \end{bmatrix} \right)$ with itself; the result is $8.7488\bar{8}$

6. Let $M = \begin{bmatrix} -15 & 22 & -11 \\ 44 & -92 & 44 \\ 110 & -220 & 106 \end{bmatrix}$

6a) Let $\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix}$. Find $M\mathbf{a}$. Is $\mathbf{a}$ an eigenvector for M? If so, what's the corresponding eigenvalue?

$$M\vec{a} = \begin{bmatrix} -7 \\ 28 \\ 70 \end{bmatrix} = 7\begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix} = 7\vec{a}$$

so yes, $\vec{a}$ is an eigenvector for M and $\lambda = 7$ is the eigenvalue.

6b) It turns out that -4 is an eigenvalue for M. Find a basis for the corresponding eigenspace.

this eigenspace is $Nul(M - -4I_3)$

$$= Nul\left(\begin{bmatrix} -11 & 22 & -11 \\ 44 & -88 & 44 \\ 110 & -220 & 110 \end{bmatrix}\right)$$

$= Nul($ rref of this $)$

$$= Nul\left(\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{ \vec{x} \;\middle|\; \vec{x} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ where } x_2 \text{ and } x_3 \text{ are free} \right\}$$

$\therefore$ a basis is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

6c) Find the characteristic polynomial of M (this should be easy based on the previous two parts).

it's $(\lambda - 7)(\lambda - -4)^2 = (\lambda - 7)(\lambda + 4)^2$

6d) Find, if possible, P and D for which $M = PDP^{-1}$, and D is a diagonal matrix whose entries are eigenvalues of M and the corresponding columns of P are eigenvectors corresponding to those eigenvalues.

Let $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$; then $P = \begin{bmatrix} -1 & 2 & -1 \\ 4 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 9 & -4 \\ -10 & 20 & -9 \end{bmatrix}$

(there are other "orderings" for the diagonal of D, and given D there are lots of ways to make P...)

6e) Use your calculator to find P^{-1}.

(see above)

7. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$ (same as in problem 4). Let R be the rref of A. Find each of the following. *Show any relevant matrices* you used in your computations:

7A) Find a basis for col(A). Call this basis $\mathcal{B}$.

First, note that $R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so the pivot columns of A form a basis and from R those cols are #'s 1 and 2 $\therefore$ a basis of col(A) is

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

7B) Find a basis for col(R).

its own pivot columns will form a basis. $\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

7C) Find a basis for row(A).

method 1: since row(A) = row(R), a basis of one will be a basis of the other. But the <u>nonzero</u> rows of a rref-matrix are a basis of its row space.
$\therefore \{[1, 0, 2], [0, 1, -3]\}$ (where we are writing the vectors sideways)

method 2: since row$(A) = $ col(A^T), they share bases. Now, rref$(A^T) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ showing the pivot cols of A^T are its 1ST and 3rd cols. $\therefore$ a basis of col(A^T) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$ and (written sideways) they are a basis of row(A)

BUT be careful: the nonzero rows of rref (A) do <u>NOT</u> point at rows which are a basis of row(A); in particular, $\{[2,1,1],[4,2,2]\}$ is <u>NOT</u> a basis here! (since row 2 is a multiple of row 1 !!!)

Problem 7, continued:

7D) Find a basis for row(R).

Since R is in rref, its non-zero rows are a basis for it.

$\therefore \{[1,0,2],[0,1,-3]\}$

(alternatively since $\text{row}(R)=\text{row}(A)=\text{col}(A^T)$,
the pivot cols of A^T will form a basis... see "method 2" on the previous problem (7C))

so another soln is $\{[2,1,1][3,1,3]\}$

NB $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is <u>NOT</u> a basis as some have claimed...}

7E) Find a basis for null(A).

$\text{null}(A)$ = the vector space of all soln's $\vec{x}$ of $A\vec{x}=\vec{0}$
which we find from $\text{rref}(A)=R$ to be

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2x_3\\3x_3\\x_3\end{bmatrix}$$ where x_3 is free

So a <u>basis</u> is $\left\{\begin{bmatrix}-2\\3\\1\end{bmatrix}\right\}$

7F) Find a basis for col$(A)^\perp$.

We need all $\vec{x}$ for which $\vec{x}\perp$ the cols of A, or
$\vec{x}\cdot(\text{each col. vector})=0$.

these same eqns appear in $A^T\vec{x}=\vec{0}$

So rref A^T to get $\text{rref}(A^T)=\begin{bmatrix}1&2&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$

showing $$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$$ where x_2 & x_4 are free.

so a basis is then $\longrightarrow \left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right\}$

8. Again, let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$ (same as in problem 4 and 7).

8A) Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Find all conditions on $b_1, \ldots b_4$ which guarantee that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$

The rref of $[A|\vec{b}]$ is represented in $\text{rref}([A|I_4]) = \left[\begin{array}{ccc|cccc} 1 & 0 & 2 & 0 & 0 & 2 & -1 \\ 0 & 1 & -3 & 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 \end{array}\right]$

which shows $A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow \begin{cases} b_1 + b_3 - b_4 = 0 \\ \quad \text{AND} \\ b_2 + 2b_3 - 2b_4 = 0 \end{cases}$

or, $b_1 = -b_3 + b_4$
$b_2 = -2b_3 + 2b_4$
where b_3 and b_4 are free

or $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

(note this says these two vectors are another basis of Col(A))

8B) Find b_1 and b_2 for which $\mathrm{w} = \begin{bmatrix} b_1 \\ b_2 \\ 13 \\ 21 \end{bmatrix}$ is in col(A). with $b_3 = 13$ and $b_4 = 21$ in the circled eqns

find $b_1 = 8$ and $b_2 = 16$

8C) Find $[\ \mathrm{w}\ ]_{\mathcal{B}}$ (see 7A, where $\mathcal{B}$ is defined, and 8B).

we need to express $\vec{w}$ as a L.C. of the elts. of that basis.

now, $\text{rref}\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 4 & 2 & 16 \\ 3 & 1 & 13 \\ 5 & 2 & 21 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{w} = 5\vec{c}_1 - 2\vec{c}_2 \Rightarrow [\vec{w}] = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

9. Suppose the determinant of some 4x4 matrix M is 5. Next to each of the following matrices, write its determinant.

M^3 125

$3M$ $3^4 \cdot 5 = 405$ "3M is M with each row multiplied by 3"; there are 4 rows.

$-M$ $(-1)^4 5 = 5$ (-M is "M with each row multiplied by -1")

$4M + 3M + 2M + M = 10M \therefore$ 50,000 $(10^4 \cdot 5)$