Math 205 A B Fall 2012 NAME (legibly!)________________________________

FINAL EXAM December 12, 2012 Circle Your SECTION: A (8 am) B (9:30 am)

DO NOT WRITE HERE!

1	
2	
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9	
TOTAL	

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!

1. Let $\mathbf{F}$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that we have been using in class. Let $H = \{f \in \mathbf{F} \mid f$ has the same y-coordinate at $x = -1$ as it does at $x = 1\}$, so a vector $\mathbf{u}$ belongs to H iff $\mathbf{u}$ is a function f satisfying $f(-1) = f(1)$.

(1A) Give an example of a *non-constant* function f belonging to H.

(1B) It's true that H a subspace of $\mathbf{F}$. For just the first TWO of the three conditions given in the three parts of the definition of a subspace (in the order we've always talked about them), prove that H satisfies that condition.

1Bi) proof that the first condition, or part, of the subspace definition holds:

1Bii) proof that the second condition of the subspace definition holds:

2. Let $\mathcal{S}$ be the vector space of all sequences $\mathbf{s} = (s_1,\ s_2,\ s_3,\ \ldots)$ of real numbers that we've discussed in class, and $\mathbf{F}$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define $T : \mathbf{F} \to \mathcal{S}$ by

$$T(f) = (f(1) + 2,\quad f(2) + 3,\quad f(3) + 4,\quad f(4) + 5,\ \ldots).$$

For example, if $f(x) = x^2$, then the *second* term in sequence $T(f)$ is $f(2) + 3 = 2^2 + 3 = 4 + 3 = 7$. The first two terms of $T(f)$ are then $(3,\ 7,\ \ldots)$.

(2A) So, if $f(x) = x^2$, what are the first *five* terms of the sequence $T(f)$?

(2B) Find $T(g)$ where $g(x) = x^3$. (Give the first five terms).

(2C) Find $T(5f) = T(5x^2)$ through the first five terms.

(2D) Is T a linear transformation? For each of the two parts of the definition of a linear transformation, either prove that T satisfies that part, or show it does not by giving a specific counterexample. I'd recommend considering the examples you've worked on in 2A–2C!

(2Di) (part one of the LT definition: your proof or counterexample):

(2Dii) (part two of the LT definition: your proof or counterexample):

3. Let $A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 2 & 4 \\ 1 & -3 & 1 \end{bmatrix}$; let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the column vectors of A, and let $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$.

(3A). Explain why $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for any $\mathbf{b} \in \mathbb{R}^3$.

(3B). Explain why the set S is linearly independent.

(3C). Parts 3A and 3B say that S is a basis for $\mathbb{R}^3$. Show that it is an orthogonal basis.

(3D.) Since S is an orthogonal basis of $\mathbb{R}^3$, there's a formula that uses dot-products to find the weights needed to express a vector $\mathbf{u} \in \mathbb{R}^3$ as a LC of the members of S. What is that formula?

(3E). Use the formula in 3D to express $\mathbf{u} = \begin{bmatrix} -333 \\ 493 \\ -109 \end{bmatrix}$ as a linear combination of the vectors in S. Show all your work. Express all weights as fractions in lowest terms.

4. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$. Let $\mathbf{y} = \begin{bmatrix} 9 \\ 18 \\ 28 \\ 4 \end{bmatrix}$.

(4A) It turns out that $\mathbf{y}$ is not in the column space of A, but you do not need to check this. Find the least squares solution of $A\mathbf{x} = \mathbf{y}$. Show all your work.

(4B) Find the projection of $\mathbf{y}$ onto the column space of A. Note well the columns are *not* orthogonal!

(4B) Find the distance from $\mathbf{y}$ to Col(A). Show all your work.

5. In this problem, in your answers and work, write all repeating decimal numbers to **4 places** after the decimal point (not fractions). For example, write 1/3 as 0.3333.... But maintain complete precision in the calculator itself.

There is no parabola passing through the points (2, 3) (3, 5) $(4,\ 3\frac{1}{6})$ $(6,\ 6\frac{1}{3})$; you do *not* need to verify this. [NOTE that $3\frac{1}{6}$ means $19/6 = 3.1666666\ldots$ and $6\frac{1}{3} = 19/3 = 6.333333\ldots$ — enter such numbers into your calculator as fractions (eg, 19/6 and 19/3) to maintain maximum precision].

(5A) Find the parabola $y = \beta_2 x^2 + \beta_1 x + \beta_0$ that is the "best-fit" parabola for these points. Show all your work, including all matrices and vectors involved in solving this problem. Remember: write just **4 places** for repeating decimal expansions.

(5B) What are the predicted values, that is, the y-coordinates of this best-fit parabola at $x = 2,\ 3,\ 4$ and 6, respectively?

(5C) What is the sum-of-the-squares ("SOS") of the residuals for the predicted values found in (5B)? Show all your computations.

(5D) A polynomial whose coefficients are "sort of close" to the best-fit coefficients is $y = 0.2x^3 - 0.5x + 4$. If this polynomial were used to find the predicted values, what would they be?

(5E) What is the SOS of the residuals for the predicted values in 5D?

6. Let $M = \begin{bmatrix} -15 & 22 & -11 \\ 44 & -92 & 44 \\ 110 & -220 & 106 \end{bmatrix}$

6a) Let $\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix}$. Find $M\mathbf{a}$. Is $\mathbf{a}$ an eigenvector for M? If so, what's the corresponding eigenvalue?

6b) It turns out that -4 is an eigenvalue for M. Find a basis for the corresponding eigenspace.

6c) Find the characteristic polynomial of M (this should be easy based on the previous two parts).

6d) Find, if possible, P and D for which $M = PDP^{-1}$, and D is a diagonal matrix whose entries are eigenvalues of M and the corresponding columns of P are eigenvectors corresponding to those eigenvalues.

6e) Use your calculator to find P^{-1}.

7. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$ (same as in problem 4). Let R be the rref of A. Find each of the following. *Show any relevant matrices* you used in your computations:

7A) Find a basis for col(A). Call this basis $\mathcal{B}$.

7B) Find a basis for col(R).

7C) Find a basis for row(A).

Problem 7, continued:

7D) Find a basis for row(R).

7E) Find a basis for null(A).

7F) Find a basis for col$(A)^{\perp}$.

8. Again, let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix}$ (same as in problem 4 and 7).

8A) Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Find all conditions on $b_1, \ldots b_4$ which guarantee that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$

8B) Find b_1 and b_2 for which $\mathbf{w} = \begin{bmatrix} b_1 \\ b_2 \\ 13 \\ 21 \end{bmatrix}$ is in col(A).

8C) Find $\left[\ \mathbf{w}\ \right]_{\mathcal{B}}$ (see 7A, where $\mathcal{B}$ is defined, and 8B).

9. Suppose the determinant of some 4x4 matrix M is 5. Next to each of the following matrices, write its determinant.

M^3

$3M$

$-M$

$4M + 3M + 2M + M$