Math 105: Review for Final Exam, Part II - SOLUTIONS

1. Consider the function \( f(x) = x^3 \ln x \) on the interval \([1/e, e^2]\).

(a) Find the \( x \)- and \( y \)-coordinates of any and all local extrema and classify each as a local maximum or local minimum.

\[
f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x} - x^2(3 \ln x + 1)
\]

\[0 = x^2(3 \ln x + 1)
\]

\[\Rightarrow x^2 = 0 \quad \text{(not in our domain)} \quad \text{or} \quad \ln x = -1/3, \quad \text{which means} \quad x = e^{-1/3}
\]

\[
\begin{array}{c|c|c}
0 < x < e^{-1/3} & e^{-1/3} < x \\
\hline
f' & negative & positive
\end{array}
\]

\[y\text{-value:} \quad f(e^{-1/3}) = (e^{-1/3})^3 \ln(e^{-1/3}) = (e^{-1})(-1/3) = (-1/3)e
\]

So, \( f \) has a local minimum at \((e^{-1/3}, -1/3e)\).

(b) Find the \( x \)- and \( y \)-coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the \( y \)-values at the local extrema and the endpoints.

\[f(1/e) = (1/e)^3 \ln(1/e) = (1/e^3)(-1) = (-1/e^3)
\]

\[f(e^{-1/3}) = (-1/3e) \quad \text{from above}
\]

\[f(e^2) = (e^2)^3 \ln(e^2) = (e^6)(2) = 2e^6
\]

So, \( f \) has a global minimum at \((e^{-1/3}, -1/3)\) and a global maximum at \((e^2, 2e^6)\).

(c) Find the \( x \)-coordinate(s) of any and all inflection points.

\[
f''(x) = 2x(3 \ln x + 1) + x^2(3 \cdot \frac{1}{x} + 0)
\]

\[0 = 6x \ln x + 2x + 3x
\]

\[0 = x(6 \ln x + 5)
\]

\[\Rightarrow x = 0 \quad \text{(not in our domain)} \quad \text{or} \quad \ln x = -5/6, \quad \text{which means} \quad x = e^{-5/6}
\]

\[
\begin{array}{c|c|c}
0 < x < e^{-5/6} & e^{-5/6} < x \\
\hline
f'' & negative & positive
\end{array}
\]

So, the \( x \)-value of the inflection point of \( f \) is \( x = e^{-5/6} \).

2. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

We know \( \frac{db}{dt} \), and we want to find \( \frac{da}{dt} \).

So, we write an equation that relates \( a \) and \( b \) and then differentiate implicitly with respect to time \( t \).

\[a^2 + 6^2 = b^2
\]

\[2a \frac{da}{dt} + 0 = 2b \frac{db}{dt}
\]

\[\frac{da}{dt} = \frac{b \frac{db}{dt}}{a}
\]
At the moment in question, \( b = 10, \ a = 8 \) (by the Pythagorean Theorem), and \( \frac{db}{dt} = -3 \).

So, \( \frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75 \) feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

3. **Use the Intermediate Value Theorem to show that** \( f(x) = x^3 - 2x - 1 \) **has a root on** \([1, 2]\).  

**IVT:** If \( f \) is continuous on \([a,\,b]\) and \( y \) is a number between \( f(a) \) and \( f(b) \), then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y \).

For the function given above, \( f(1) = -2 \) and \( f(2) = 3 \). Since 0 is a number between \(-2\) and 3, the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0 \); this \( c \) is the desired root.

4. **Use Newton’s Method with an initial guess of** \( x_0 = 1.5 \) **to find the next three approximations to a solution of** \( x^3 - 2x - 1 = 0 \). **Then test your final approximation to see if it appears to be close to a root.**

Recall that \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 1}{3x_n^2 - 2} \).

Using a calculator with initial guess \( x_0 = 1.5 \), we get the following.

\[
\begin{align*}
    x_1 &= 1.63157... \\
    x_2 &= 1.61818... \\
    x_3 &= 1.61803...
\end{align*}
\]

Check: \((1.61803)^3 - 2(1.61803) - 1 \approx -0.0000233\), so our approximation seems to be a good one.

5. **What (if anything) does the Extreme Value Theorem say about** \( f(x) = x^2 \) **on each of the following intervals?**

**EVT:** If \( f \) is continuous on \([a,\,b]\), then \( f \) has both a maximum and a minimum on \([a,\,b]\).

(a) \([1, 4]\]

\( f \) has a maximum and a minimum on \([1, 4]\)

(b) \((1, 4)\)

The EVT doesn’t apply because \((1, 4)\) is not a closed interval since its endpoints are not included.

6. **Find the value of the constant** \( c \) **that the Mean Value Theorem specifies for** \( f(x) = x^3 + x \) **on** \([0, 3]\).

**MVT:** If \( f \) is continuous on \([a,\,b]\) and differentiable on \((a,\,b)\), then there is a number \( c \) between \( a \) and \( b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

For our function, we have \( \frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10 \).

And \( f'(x) = 3x^2 + 1 \), so \( f'(c) = 3c^2 + 1 \).

So, we solve \( 3c^2 + 1 = 10 \), which means \( c = \sqrt{3} \).

7. **Find the following.**

(a) **all antiderivatives of** \( 1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5} \)

Any such antiderivative will take the form \( x + x^2 + \frac{x^4}{4} + \frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + C \).

Note that we have used the facts that \( \sqrt{x} = x^{1/2} \) and \( 1/x^5 = x^{-5} \).

(b) \( \int_1^7 \frac{3}{x} \, dx = 3 \ln |x| \bigg|_1^7 = 3 \ln 7 - 3 \ln 1 = 3 \ln 7 \)
(c) \( \int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi \)  
This integral represents the area of a semicircle of radius 2.

(d) \( \frac{d}{dx} \int_{1}^{x} \sin \sqrt{t} \, dt = \sin \sqrt{x} \)  
The derivative of the area function is the original function.

(e) \( \lim_{n \to \infty} \frac{2}{n} \sum_{k=1}^{n} \left(1 + \frac{2k}{n}\right)^2 \)  
[The 8:00 and 9:30 sections may omit this part.]

This represents the limit of a right-hand sum as the number \( n \) of rectangles goes to infinity.

As \( k \) goes from 1 to \( n \), the expression \( \left(1 + \frac{2k}{n}\right)^2 \) takes on the values \( \left(1 + \frac{2}{n}\right)^2, \left(1 + \frac{4}{n}\right)^2, \ldots, \left(1 + \frac{2n}{n}\right)^2 \); the first of these values is just to the left of 1 and the last is equal to 3, so we see that we are looking at the function on the interval \([1, 3]\).

The \( \frac{2}{n} \) out front is our \( \Delta x \), which confirms that we are dealing with an interval of length 2 being subdivided into \( n \) equal subintervals.

Finally, each of the \( x \)-values is being squared, so the function in question must be \( f(x) = x^2 \).

Thus, we see that the expression is the area under \( f(x) = x^2 \) on \([1, 3]\).

Its value is \( \int_{1}^{3} x^2 \, dx = \frac{x^3}{3} \bigg|_{1}^{3} = \frac{26}{3} \).

8. Water is leaking out of a tank at a decreasing rate \( r(t) \) as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

overestimate = \( L_4 = (15 + 11 + 8 + 4)(2) = 76 \) 
underestimate = \( R_4 = (11 + 8 + 4 + 3)(2) = 52 \)

(b) Interpret the expression \( \int_{2}^{6} r(t) \, dt \) in terms of the situation described above.

This integral gives the amount of water (in gallons) that leaked out of the tank on the interval \([2, 6]\) minutes.
9. Consider the graph of \( f(t) \) shown. It is made of straight lines and a semicircle.

Let \( G(x) = \int_0^x f(t) \, dt \) and \( H(x) = \int_{-3}^x f(t) \, dt \).

(a) Compute \( G(2) \), \( G(4) \), and \( H(4) \).

\( G(2) \) is the area under \( f \) between \( t = 0 \) and \( t = 2 \). This is a rectangle plus a triangle and has area \( 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3 \).

Similarly, \( G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 3 + \frac{\pi}{2} \).

\( H(4) \) is the area under \( f \) between \( t = -3 \) and \( t = 4 \). Remember that area below the \( t \)-axis counts as negative.

\[
H(4) = - (2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1) + \frac{1}{2} \cdot 2 \cdot 1 + \text{[area under } f \text{ from 0 to 4, found above as } G(4)]
\]
\[
= -2 + \left[ 3 + \frac{\pi}{2} \right]
\]
\[
= 1 + \frac{\pi}{2}
\]

(b) Where is \( G \) increasing? Where is \( G \) decreasing?

\( G \) is increasing where \( f \) is positive: \((-1, 4]\). Note that \( G \) has a horizontal slope at \( x = 2 \) but since \( f \) is positive on each side of \( t = 2 \), we say \( G \) is increasing at \( x = 2 \).

\( G \) is decreasing where \( f \) is negative: \([-4, -1) \).

(c) Where is \( G \) concave up? Where is \( G \) concave down?

\( G \) is concave up where \( f \) is increasing: \((-2, 0) \cup (2, 3) \).
\( G \) is concave down where \( f \) is decreasing: \((1, 2) \cup (3, 4) \).

(d) At what \( x \)-value(s) does \( G \) have a local maximum? At what \( x \)-value(s) does \( G \) have a local minimum?

\( G \) has a local maximum where \( f \) changes from positive to negative: never.
\( G \) has a local minimum where \( f \) changes from negative to positive: \( x = -1 \).

(e) Find a formula that relates \( G \) and \( H \).

From their definitions, \( H(x) = \int_{-3}^x f(t) \, dt + G(x) = -2 + G(x) \).

(f) How would your answers to (b), (c), and (d) change if the questions were about \( H \) instead of \( G \)?

They would not change at all because \( H'(x) = G'(x) \).
10. (a) Use sigma notation to express \( L_{10} \) and \( M_{10} \) as approximations to \( \int_{20}^{60} \ln x \, dx \).

We’re subdividing the interval into 10 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{10} = 4 \).

\[
L_{10} = \left[ f(20) + f(24) + f(28) + \ldots + f(52) + f(56) \right] \Delta x = \left[ \ln(20) + \ln(24) + \ln(28) + \ldots + \ln(52) + \ln(56) \right] \cdot 4 
\]

\[
M_{10} = \left[ f(22) + f(26) + f(30) + \ldots + f(54) + f(58) \right] \Delta x = \left[ \ln(22) + \ln(26) + \ln(30) + \ldots + \ln(54) + \ln(58) \right] \cdot 4 
\]

(b) Draw a sketch that represents the sum \( M_4 \).

Now we’re subdividing the interval into 4 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{4} = 10 \).

Note that the height of each rectangle is determined by the \( y \)-value of the curve at the middle \( x \)-value of the rectangle (that is, at \( x = 25, 35, 45, 55 \)).

11. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is $9.00 per container, what dimensions will give the largest volume? [The 8:00 and 9:30 sections may omit this problem.]

Area of circle = \( \pi r^2 \)  
Lateral area of cylinder = \( 2\pi rh \)  
Volume of cylinder = \( \pi r^2 h \)

Objective function: \( V = \pi r^2 h \)

We need to get this down to a function of just one variable, so we use the

constraint equation: \( \text{cost} = 900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2 \pi rh \)

\[
900 = 6\pi r^2 + 10\pi rh \\
900 - 6\pi r^2 = 10\pi rh \\
\frac{900 - 6\pi r^2}{10\pi r} = h 
\]

Substituting this back into the objective function gives

\[
V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10}(900r - 6\pi r^3) .
\]
Now that we have $V$ as a function of just one variable, we find its maximum.

$$V'(x) = \frac{1}{10}(900 - 18\pi r^2)$$

$$0 = \frac{1}{10}(900 - 18\pi r^2)$$

$$\Rightarrow 18\pi r^2 = 900$$

$$\Rightarrow r^2 = \frac{50}{\pi}$$

$$\Rightarrow r = \sqrt{\frac{50}{\pi}}$$

<table>
<thead>
<tr>
<th>$f'$</th>
<th>$0 &lt; x &lt; \sqrt{\frac{50}{\pi}}$</th>
<th>$\sqrt{\frac{50}{\pi}} &lt; x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>positive</td>
<td>negative</td>
</tr>
</tbody>
</table>

Thus, we have in fact found the global maximum at $r = \sqrt{\frac{50}{\pi}}$.

And $h = \frac{900 - 6\pi r^2}{10\pi r}$ = ...much simplifying... = $\sqrt{\frac{72}{\pi}} \approx 4.787$ inches.