Math 106 Fall 2007

Final Exam (100 points)

Name: ____

Show all your work to receive full credit for a problem.

Do not use the calculator integral function or any other programs on your calculator.

You may use formulas 1-18, 40-42, 50, 51 only from the table of integrals for any integral problem. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Round off your answers to four decimal places.

Include units in your answers wherever possible.

There are fourteen questions on six pages. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\, dx$

$$\int u\, dv = uv - \int v\, du$$

$$|I - L_n| \leq \frac{K_1(b-a)^2}{2n} \qquad |I - R_n| \leq \frac{K_1(b-a)^2}{2n}$$

$$|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} \qquad |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$$

$$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}\, |x-x_0|^{n+1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}\, s} exp\left(\frac{-(x-m)^2}{2s^2}\right)$$

$\int_1^\infty \frac{1}{x^p}\, dx$ converges for $p > 1$ and diverges for $p \leq 1$.

$\int_0^1 \frac{1}{x^p}\, dx$ converges for $p < 1$ and diverges for $p \geq 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for x in $(-\infty, \infty)$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for x in $(-\infty, \infty)$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for x in $(-\infty, \infty)$.

1. (8 points) Solve the following differential equation.

$$\sqrt{9+x^2}\,\frac{dy}{dx} = x^2y^2\,.$$

2. (9 points) Evaluate the following integral exactly.

$$\int \frac{2x^2 + x - 3}{x(x^2+1)}\, dx$$

3. (7 points) The growth of an animal population is governed by the differential equation $\frac{dP}{dt} = P(0.1 - 0.001P)$ animals/year. If $P(0) = 200$, use Euler's method with two steps to estimate the population when $t = 2$ years.

4. (7 points) The annual rainfall in a certain city is normally distributed with a mean of 15 inches and a standard deviation of 1 inch.

   (a) Write the integral that gives the fraction of the years with rainfall between 13 and 17 inches.

   (b) Evaluate the integral in part(a) by doing a u-substitution and then using the table on the first page of the exam.

5. (6 points) A tank in the shape of a right circular cone of height 13 ft and radius 2 ft is buried completely underground with its vertex 6 ft below the ground level. If the tank is filled with water (density = 62.4 lb/cubic foot) to a height of 9 ft, write (but do not evaluate) an integral equal to the work done in pumping all the water in the tank to ground level.

6. (8 points) Consider the region bounded by the curve $y = e^{(x^2)}$, and the lines $x = 0$, $x = 2$ and the x-axis.

(a) Write (but do not evaluate) an integral to find the volume of the solid that is formed when the region is rotated about the x-axis.

(b) What is the least value of n which guarantees that a left sum approximation L_n approximates the integral in part (a) within ± 0.1? Justify your answer.

7. (6 points) Let $\{S_n\}$ denote the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$. In each of the following cases, say whether the series must converge, may converge or does not converge. If the series must converge, find the sum of the series.

   (a) $\lim_{n\to\infty} a_n = 0$

   (b) $\lim_{n\to\infty} S_n = 0$

   (c) $\lim_{n\to\infty} S_n = 2$

8. (7 points) Use comparisons to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=1}^{\infty} \frac{4}{3^n + 5n + 2}.$$

9. (8 points) Determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{k=1}^{\infty} ke^{-2k}$$

10. (5 points) Determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=3}^{\infty} n \ln n$$

11. (8 points) Find the interval and radius of convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{k(x-3)^k}{(2k)!}$.

12. (9 points) Let $f(x) = \sin(2x^3)$. Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for f.

(b) Use the series in part (a) to find $f^{(225)}(0)$ and $f^{(230)}(0)$.

13. (8 points) Let $f(x) = \dfrac{1}{x}$. Find the first four non-zero terms of the Taylor series for f based at $x_0 = 1$. Then write the series using the summation notation.

14. (4 points) Find the exact sum of the series $1 - 2 + \dfrac{4}{2!} - \dfrac{8}{3!} + \ldots$.