Math 105D - Final Exam - December 12, 2006

Instructions: Show all of your work and circle your final answers. Calculators are allowed, but notes and books are not.

1. (a) (5 points) Evaluate \( \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \).

\[
\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{x}{2 \ln x} \cdot \frac{1}{\sqrt{x}} = \lim_{x \to \infty} \frac{x}{2 \ln x} \cdot \frac{1}{\sqrt{x}} = \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \infty.
\]

(b) (5 points) Simplify \( \tan(\arcsin 3x) \).

Let \( \theta = \arcsin 3x \), so \( \sin \theta = 3x = \frac{opp}{hyp} \).

\[
\begin{align*}
\theta & \quad 3x \quad y = \sqrt{1-9x^2} \\
1 & \quad \quad y = \sqrt{1-9x^2} \quad y = \sqrt{1-9x^2}
\end{align*}
\]

So \( \tan(\arcsin (3x)) = \tan \theta = \frac{opp}{adj} = \frac{3x}{\sqrt{1-9x^2}} \).
(c) (6 points) Find \( \frac{d}{dx} (x \arctan 2x)(\ln x) + \cos (e^{x^2} + 3) \).

Note: \( \frac{d}{dx} \left( f(x)g(x)h(x) \right) = f'(x)g(x)h(x) + f(x)g(x)h'(x) + f(x)g'(x)h(x) \).

Product Rule twice

\[
\begin{align*}
\frac{d}{dx} \left( (x \arctan 2x)(\ln x) + \cos (e^{x^2} + 3) \right) &= f'(x)g(x)h(x) + f(x)g'(x)h(x) \\
&= \left( 1 \cdot \arctan 2x \right) \ln x + \left( x \cdot \frac{2}{1+2^2} \right) \cdot \ln x + x \arctan (2x) \cdot \frac{1}{x} \\
&\quad + \sin (e^{x^2} + 3) \cdot (e^{x^2} + 2x)
\end{align*}
\]

2. (8 points) Consider the function \( f(x) = \frac{2}{3x} \). Use the limit definition of the derivative to find \( f'(x) \).

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{2}{3(x+h)} - \frac{2}{3x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{2(3x) - 2(3(x+h))}{3(x+h) \cdot 3x}}{h} \\
&= \lim_{h \to 0} \frac{6x - 6x - 6h}{h \cdot 9(x+h) \cdot x} \\
&= \lim_{h \to 0} \frac{-6h}{h \cdot 9(x+h) \cdot x} \\
&= \lim_{h \to 0} \frac{-6}{9(x+h) \cdot x} \\
&= \frac{-6}{9(x+0) \cdot x} = \frac{-6}{9x^2} = \frac{-2}{3x^2}
\end{align*}
\]
3. The graph of \( y = f(t) \) is given below. Consider the area function \( A(x) = \int_{2}^{x} f(t) \, dt \).

\[ \frac{\text{d}}{\text{d}x} (A(x)) = f(x) \]
\[ \text{i.e., } A'(x) = f(x) \]

(a) (3 points) At what \( x \)-value(s) does \( A \) have a stationary point?

\[ A'(x) = 0 \text{ when } f(x) = 0. \to x = -3.5, -2, 2 \]

(b) (3 points) At what \( x \)-value(s) does \( A \) have an inflection point?

\[ A(x) \text{ has an inflection pt. when } A'(x) \text{ changes from increasing to decreasing.} \]
\[ A'(x) = f(x), \text{ so at } x = -3, -2, 1, 3. \]

(c) (3 points) On what interval(s) is \( A \) increasing?

\[ A(x) \text{ increases when } A'(x) > 0, \text{ so when } f(x) > 0. \to (-3.5, 2) \]

(d) (3 points) On what interval(s) is \( A \) concave down?

\[ A(x) \text{ CD when } A'(x) \text{ is decreasing, so when } f(x) \text{ is decreasing.} \to (-3, -2), (1, 3) \]

(e) (6 points) Find the equation of the tangent line to the graph of \( y = A(x) \) at \( x = 4 \).

\[ A(4) = \int_{2}^{4} f(t) \, dt \]
\[ A'(4) = f(4) = -2. \]
\[ p_t = (4, -4), \text{ slope } = -2. \]
\[ y - (-4) = -2(x - 4) \]
\[ y + 4 = -2x + 8 \]
\[ \boxed{y = -2x - 6 \text{ is the equation of the tan line at } x = 4.} \]
4. (12 points) At a certain moment, Car A is 4 miles east of the intersection traveling toward the intersection at a rate of 50 miles/hour. At the same time, Car B is 3 miles south of the intersection traveling away from the intersection at a rate of 60 miles/hour. Is the distance between the cars increasing or decreasing at that moment? At what rate?

Let \( a = \text{dist from Car A to intersection} \)
\( b = \text{dist from Car B to intersection} \)
\( c = \text{dist from Car A to Car B} \).

\( a, b, \) and \( c \) are functions of time, so we can consider \( \frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt} \).

\( a \) is decreasing, so \( \frac{da}{dt} = -50 \text{ mph} \).
\( b \) is increasing, so \( \frac{db}{dt} = +60 \text{ mph} \).

We want to find \( \frac{dc}{dt} \) when \( a = 4 \) and \( b = 3 \).

at the moment when

Pythagorean Theorem: \( a^2 + b^2 = c^2 \).

Differentiate w.r.t. \( t \):
\[
\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt}(c^2)
\]
So
\[
2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}.
\]
Plug in known quantities:
\[
2(4)(-50) + 2(3)(60) = 2c \frac{dc}{dt}. \\
-400 + 360 = 2c \frac{dc}{dt}.
\]
\[
-40 = 2c \frac{dc}{dt}.
\]
\[
-20 = c \frac{dc}{dt}. \\
\]
\[
-\frac{20}{c} = \frac{dc}{dt}.
\]

When \( a = 4, b = 3, \ c = a^2 + b^2 = 4^2 + 3^2 = 25 \),
\( c = \pm 5 \), so \( c = 5 \) (don't have negative dist.).

So \( \frac{dc}{dt} = -\frac{20}{5} = -4 \text{ mph. Hence, the distance is decreasing.} \)
5. (8 points) The average value of a function $f$ on the interval $[0,3]$ is 4, and the average value of $f$ on the interval $[0,5]$ is 6. What is the average value of $f$ on the interval $[3,5]$?

Average value of $f$ on $[a,b]$ is \( \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \).

Average value of $f$ on $[0,3]$ is 4 \( \rightarrow \frac{1}{3-0} \int_{0}^{3} f(x) \, dx = 4 \), so \( \frac{1}{3} \int_{0}^{3} f(x) \, dx = 4 \),

\( \frac{3}{5} \int_{0}^{5} f(x) \, dx = 12 \).

Average value of $f$ on $[0,5]$ is 6 \( \rightarrow \frac{1}{5-0} \int_{0}^{5} f(x) \, dx = 6 \), so \( \frac{5}{6} \int_{0}^{6} f(x) \, dx = 30 \).

Thus, the average value of $f$ on $[3,5]$ is \( \frac{1}{5-3} \int_{3}^{5} f(x) \, dx \).

And \( \int_{3}^{5} f(x) \, dx = \int_{0}^{5} f(x) \, dx - \int_{0}^{3} f(x) \, dx = 30 - 12 = 18 \).

So, average value of $f$ on $[3,5]$ is \( \frac{1}{2} (18) = 9 \).

6. (9 points) Consider the differential equation $y' - 2y = \frac{y}{t}$. Is $y(t) = 5te^{2t}$ a solution?

If $y(t) = 5te^{2t}$, $y'(t) = 5e^{2t} + 5t(2e^{2t}) = 5e^{2t} + 10te^{2t}$.

So, $y'(t) - 2y(t) = 5e^{2t} + 10te^{2t} - 2(5te^{2t})$

$= 5e^{2t} + 10te^{2t} - 10te^{2t} = 5e^{2t}$.

And \( \frac{y}{t} = \frac{5te^{2t}}{t} = 5e^{2t} \).

So, $y' - 2y = \frac{y}{t}$. Thus, we have a solution.
7. (14 points)

(a) What are the hypotheses of the intermediate value theorem?

If the function \( f(x) \) is continuous on a closed interval \([a, b]\), ...

(b) What does the intermediate value theorem say about the function \( g(x) = x^4 + x \) on the interval \([0, 2]\)?

\( g(x) \) is continuous on \([0, 2]\), a closed interval, so the IVT says that for every \( y \) between \( g(0) \) and \( g(2) \) (i.e., between 0 and 18), there is some \( x \)-value \( c \) in \([0, 2]\) where \( g(c) = y \).

(c) What are the hypotheses of the mean value theorem?

If the function \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), ...

(d) What does the mean value theorem say about the function \( g(x) = x^4 + x \) on the interval \([0, 2]\)? Find all suitable values of \( c \).

\( g(x) = x^4 + x \) is continuous on \([0, 2]\) and differentiable on \((0, 2)\), so the MVT says there is an \( x \)-value \( c \) where:

\[
| \frac{g(b) - g(a)}{b - a} = \frac{g(b) - g(a)}{b - a} = \frac{18 - 0}{2} = 9.
\]

\( g'(x) = 4x^3 + 1 \), so \( g'(c) = 4c^3 + 1 = 9 \).

\[
4c^3 = 8 \quad \Rightarrow \quad c = \sqrt[3]{2}. \]
8. (15 points) Consider the integral \( \int_0^1 (x^2 + 1) \, dx \).

(a) Use the Fundamental Theorem of Calculus to evaluate this integral exactly.

\[
\int_0^1 x^2 + 1 \, dx = \left. \frac{1}{3} x^3 + x \right|_0^1 = \left( \frac{1}{3} (1)^3 + 1 \right) - \left( \frac{1}{3} (0)^3 + 0 \right) \\
= \frac{1}{3} + 1 - 0 \\
= \frac{4}{3}
\]

(b) Approximate the integral using right endpoints with three subintervals.

\[
R_3 = \int_{1/3}^{1} f(x) \, dx \\
\Delta x = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3}
\]

\[
\int_{1/3}^{1} f(x) \, dx = \int_{1/3}^{1} \left( \left( \frac{1}{3} \right)^2 + 1 \right) \, dx + \int_{1/3}^{1} \left( \left( \frac{2}{3} \right)^2 + 1 \right) \, dx + \int_{1/3}^{1} \left( \left( \frac{3}{3} \right)^2 + 1 \right) \, dx
\]

\[
= \frac{1}{3} \left( \frac{10}{9} + \frac{13}{9} + 2 \right) = \frac{1}{3} \left( \frac{10}{9} + \frac{13}{9} + \frac{18}{9} \right)
\]

\[
= \frac{41}{27}
\]

(c) Write your answer to part (c) in summation notation (i.e., using \( \Sigma \)).

\[
R_3 = \sum_{k=1}^{3} f \left( \frac{k}{3} \right) \Delta x = \sum_{k=1}^{3} \left( \left( \frac{k}{3} \right)^2 + 1 \right) \cdot \frac{1}{3}
\]
(Problem 8 continued - still working with $\int_0^1 (x^2 + 1) \, dx$)

(d) Use summation notation to express the approximation of the integral with right endpoints on $n$ subintervals.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$R_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \Delta x = \sum_{k=1}^{n} \left(\left(\frac{k}{n}\right)^2 + 1\right) \cdot \frac{1}{n}$$

(e) Would a Riemann sum with left endpoints on 50 subintervals give an underestimate, an overestimate, or can we not be sure? Explain. (A picture may be helpful. Note: You do not have to calculate $L_{50}$.)

Since $f(x) = x^2 + 1$ is increasing, the areas of the rectangles using left endpoints will underestimate the actual area.