Math 105D - Final Exam - December 12, 2006

Instructions: Show all of your work and circle your final answers. Calculators are allowed, but notes and books are not.

1. (a) (5 points) Evaluate \( \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \).

(b) (5 points) Simplify \( \tan(\arcsin 3x) \).
(c) (6 points) Find \( \frac{d}{dx} \left( x(\arctan 2x)(\ln x) + \cos (e^{x^2} + 3) \right) \).

2. (8 points) Consider the function \( f(x) = \frac{2}{3x} \). Use the limit definition of the derivative to find \( f'(x) \).
3. The graph of \( y = f(t) \) is given below. Consider the area function \( A(x) = \int_{2}^{x} f(t) \, dt \).

(a) (3 points) At what \( x \)-value(s) does \( A \) have a stationary point?

(b) (3 points) At what \( x \)-value(s) does \( A \) have an inflection point?

(c) (3 points) On what interval(s) is \( A \) increasing?

(d) (3 points) On what interval(s) is \( A \) concave down?

(e) (6 points) Find the equation of the tangent line to the graph of \( y = A(x) \) at \( x = 4 \).
4. (12 points) At a certain moment, Car A is 4 miles east of an intersection traveling toward the intersection at a rate of 50 miles/hour. At the same time, Car B is 3 miles south of the intersection traveling away from the intersection at a rate of 60 miles/hour. Is the distance between the cars increasing or decreasing at that moment? At what rate?
5. (8 points) The average value of a function $f$ on the interval $[0,3]$ is 4, and the average value of $f$ on the interval $[0,5]$ is 6. What is the average value of $f$ on the interval $[3,5]$?

6. (9 points) Consider the differential equation $y' - 2y = \frac{y}{t}$. Is $y(t) = 5te^{2t}$ a solution?
7. (14 points)

(a) What are the hypotheses of the intermediate value theorem?

(b) What does the intermediate value theorem say about the function $g(x) = x^4 + x$ on the interval $[0, 2]$?

(c) What are the hypotheses of the mean value theorem?

(d) What does the mean value theorem say about the function $g(x) = x^4 + x$ on the interval $[0, 2]$? Find all suitable values of $c$. 
8. (15 points) Consider the integral \( \int_0^1 (x^2 + 1) \, dx \).

(a) Use the Fundamental Theorem of Calculus to evaluate this integral exactly.

(b) Approximate the integral using right endpoints with three subintervals.

(c) Write your answer to part (c) in summation notation (i.e., using \( \sum \)).
(Problem 8 continued - still working with $\int_{0}^{1} (x^2 + 1) \, dx$.)

(d) Use summation notation to express the approximation of the integral with right endpoints on $n$ subintervals.

(c) Would a Riemann sum with left endpoints on 50 subintervals give an underestimate, an overestimate, or can we not be sure? Explain. (A picture may be helpful. Note: You do not have to calculate $L_{50}$.)