Math 205B Final Exam (100 points)

Name: Solutions.

- Check that you have 8 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (15 points) Let $W$ be the subspace spanned by the two vectors $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

   Let $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

   (a) Is the vector $\vec{y}$ in $W$? Explain.

   $$[\bar{u}_1 \quad \bar{u}_2 \quad \bar{y}] \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The last row indicates that the system is inconsistent.

   So $\bar{y}$ cannot be written as a linear combination of $\bar{u}_1$ and $\bar{u}_2$. So $\bar{y}$ is not in $W$.

   (b) Find a vector in $W$ that is closest to $\vec{y}$.

   $$\hat{y} = \left(\frac{\bar{u}_1 \cdot \bar{y}}{\bar{u}_1 \cdot \bar{u}_1}\right)\bar{u}_1 + \left(\frac{\bar{u}_2 \cdot \bar{y}}{\bar{u}_2 \cdot \bar{u}_2}\right)\bar{u}_2$$

   $$= \frac{7}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

   $$= \begin{bmatrix} 4/3 \\ 19/6 \\ 7/6 \end{bmatrix}$$

   (c) Find a vector that is orthogonal to $W$.

   $$\bar{z} = \bar{y} - \hat{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 19/6 \\ 7/6 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/6 \\ 5/6 \end{bmatrix}.$$

2. (15 points) Orthogonally diagonalize the matrix $A = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$. (The eigenvalues of $A$ are 25 and $-25$.)

$$A - 25I = \begin{bmatrix} -32 & 24 \\ 24 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = \frac{3}{4} x_2 \\ x_2 \text{ free.} \end{array}$$

Choosing. $x_2 = 4$, we get a basis for the eigenspace corresponding to 25 which is $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

$$A + 25I = \begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = -4/3\, x_2 \\ x_2 \text{ free.} \end{array}$$

Choosing $x_2 = 3$, we get a basis for the eigenspace corresponding to $-25$ which is $\left\{ \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$.

So we can diagonalize A by choosing

$$P = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}.$$

To orthogonally diagonalize. A, make the columns of P an orthonormal set. The columns are already orthogonal to each other. So simply normalize them.

Length of first column $= \sqrt{3^2 + 4^2} = 5$.

Length of second column $= 5$ as well.

So we can orthogonally diagonalize A by choosing $Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}, \quad D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}$.

3. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace.

(a) $W = \left\{ \begin{bmatrix} 2a - c + d \\ b - 2c - 2d \\ a + 3b + d \\ 2b + c + d \end{bmatrix} : a, b, c, d \text{ are real numbers.} \right\}$.

$$\begin{bmatrix} 2a-c+d \\ b-2c-2d \\ a+3b+d \\ 2b+c+d \end{bmatrix} = a\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

So $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ and hence W is a subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & -2 & -2 \\ 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three vectors form a linearly independent subset of the spanning set.

So basis for $W = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\dim W = 3$.

(b) All polynomials in $\mathbb{P}_3$ of the form $t + a$. Call this set $W$.

The zero polynomial cannot be written in the form $t+a$ for any real number $a$.

So the zero vector is not in $W$.

So $W$ is not a subspace of $\mathbb{P}_3$.

4. (12 points) Let $\vec{p}_1(t) = 2t - t^2$, $\vec{p}_2(t) = 2t$, $\vec{p}_3(t) = 2 - t$.

(a) Use coordinate vectors to show that $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a basis for $\mathbb{P}_2$.

With respect to the standard basis $\{1, t, t^2\}$ of $\mathbb{P}_2$, the coordinate vectors of $\bar{p}_1, \bar{p}_2, \bar{p}_3$ are

$$\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ respectively.}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & -1 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Pivot in each column $\Longrightarrow$ the set of these three vectors is a linearly independent set.

Pivot in each row $\Longrightarrow$ the set spans $\mathbb{R}^3$.

So the polynomials $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ span $\mathbb{P}_2$ and form a linearly independent set. Hence $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) Find the polynomial $\vec{q}$ in $\mathbb{P}_2$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

The coordinate vector of $\bar{q}$ is

$\cancel{0\bar{p}_1 + 1\cdot\bar{p}_2 + (-1)\bar{p}_3}$

$$0 \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}.$$

So $\bar{q} = -2 + 3t$.

5. (10 points) Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^7$.

(a) Show that $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$ is also a linearly independent set.

We show that the equation $c_1(\bar{v}_1+\bar{v}_2)+c_2(\bar{v}_1-\bar{v}_2)=\bar{0}$ has only trivial soln., i.e, the soln. $c_1=0$, $c_2=0$.

$$c_1(\bar{v}_1+\bar{v}_2)+c_2(\bar{v}_1-\bar{v}_2)=\bar{0}$$

So $(c_1+c_2)\bar{v}_1+(c_1-c_2)\bar{v}_2=\bar{0}$

Since $\{\bar{v}_1, \bar{v}_2\}$ is a linearly independent set, we have

$$\left.\begin{array}{l} c_1+c_2=0 \\ c_1-c_2=0 \end{array}\right\}$$ This set of equations implies $c_1=0$, $c_2=0$.

(b) Is $\vec{v}_1$ in Span$\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$? Explain.

$$\bar{v}_1 = \frac{1}{2}(\bar{v}_1+\bar{v}_2)+\frac{1}{2}(\bar{v}_1-\bar{v}_2).$$

So $\bar{v}_1$ is a linear combination of $\bar{v}_1+\bar{v}_2$ and $\bar{v}_1-\bar{v}_2$.

So $\bar{v}_1$ is in Span $\{\bar{v}_1+\bar{v}_2, \bar{v}_1-\bar{v}_2\}$.

(To get the values $\frac{1}{2}$, set up the eqn. $\bar{v}_1 = c_1(\bar{v}_1+\bar{v}_2)+c_2(\bar{v}_1-\bar{v}_2)$.

So $\bar{v}_1=(c_1+c_2)\bar{v}_1+(c_1-c_2)\bar{v}_2$. $\begin{array}{l} c_1-c_2=0 \\ c_1+c_2=1 \end{array}$ We get $c_1=\frac{1}{2}$, $c_2=\frac{1}{2}$.)

6. (8 points) Suppose $U$ is an $n \times n$ orthogonal matrix. For every vector $\vec{x}$ in $\mathbb{R}^n$, show that the length of the vector $U\vec{x}$ is the same as the length of the vector $\vec{x}$. (Hint: Length of $U\vec{x}$ is $\sqrt{U\vec{x} \cdot U\vec{x}}$ and length of $\vec{x}$ is $\sqrt{\vec{x} \cdot \vec{x}}$. So it is enough to show that $U\vec{x} \cdot U\vec{x} = \vec{x} \cdot \vec{x}$.)

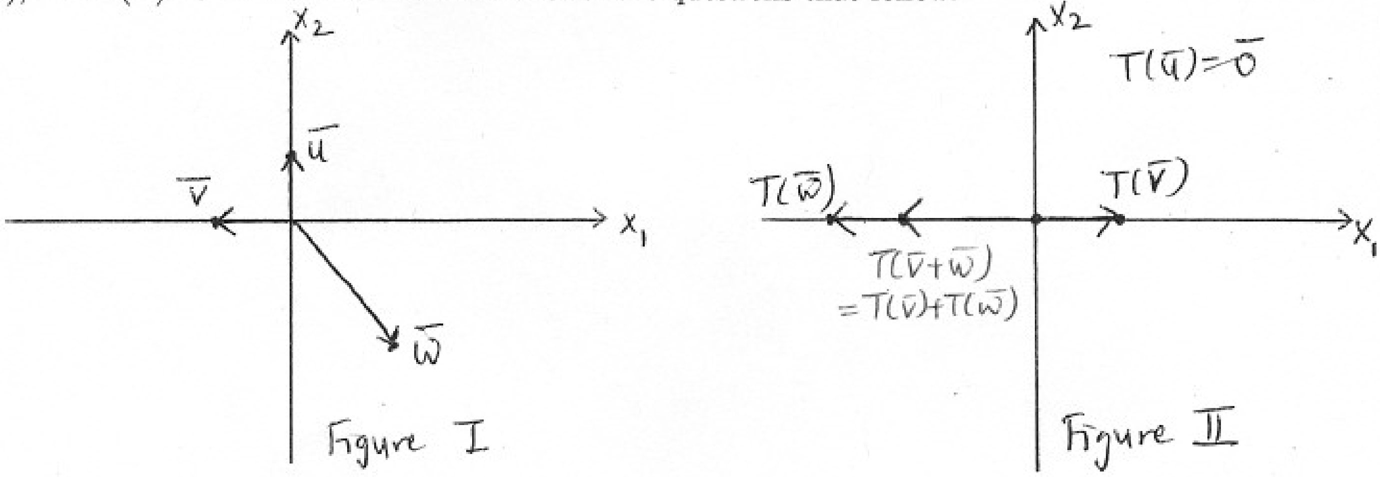
$$U\bar{x}\cdot U\bar{x} = (U\bar{x})^T U\bar{x} = \bar{x}^T U^T U \bar{x} = \bar{x}^T I_n \bar{x}$$

Since $U$ is an orthogonal matrix, $U^TU=I_n$.

So $U\bar{x}\cdot U\bar{x} = \bar{x}^T I_n \bar{x} = \bar{x}^T\bar{x} = \bar{x}\cdot\bar{x}$.

Length of $U\bar{x} = \sqrt{U\bar{x}\cdot U\bar{x}} = \sqrt{\bar{x}\cdot\bar{x}} =$ length of $\bar{x}$.

7. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ ($A$ is a $2 \times 2$ matrix). Figure I below shows vectors $\vec{u}, \vec{v}$ and $\vec{w}$ and Figure II below shows vectors $T(\vec{u})$, $T(\vec{v})$, and $T(\vec{w})$. Use this information to answer the questions that follow.




(a) In Figure II, draw $T(\vec{v} + \vec{w})$.

$$T(\bar{v} + \bar{w}) = T(\bar{v}) + T(\bar{w})$$

(b) Which of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ (if any) are eigenvectors of $A$? What are the corresponding eigenvalues? Explain.

$T(\bar{u}) = \bar{0} = 0 \cdot \bar{u}$    ie $A\bar{u} = 0\bar{u}$

So $\bar{u}$ is an eigenvector of $A$ with eigenvalue $0$.

$T(\bar{v}) = -\bar{v}$    ie $A\bar{v} = -1\bar{v}$

So $\bar{v}$ is an eigenvector of $A$ with eigenvalue $-1$.

Since $T(\bar{w})$ is not a scalar multiple of $\bar{w}$, $\bar{w}$ is not an eigenvector of $A$.

(c) Is $T$ one-to-one? Explain.

$T(\bar{u}) = \bar{0}$ and $\bar{u}$ is a non-zero vector.

So the eqn. $T(\bar{x}) = \bar{0}$ has atleast two solutions – $\bar{0}$ and $\bar{u}$.

Hence $T$ is not one-to-one.

8. (18 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

(a) Find the distance between the vector $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\bar{u} - \bar{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\|\bar{u} - \bar{v}\| = \sqrt{1+1} = \sqrt{2} = \text{distance between } \bar{u} \text{ and } \bar{v}.$$

(b) A $2 \times 4$ matrix has rank 2. Find the dimension of the null space of this matrix.

$4 = \text{rank} + \dim \text{Nul } A$ if we call the matrix $A$.

So $\dim \text{Nul } A = 4 - 2 = 2$.

(c) Suppose $A$ is a $4 \times 4$ matrix with $\det A = 20$. What is $\det 3A$?

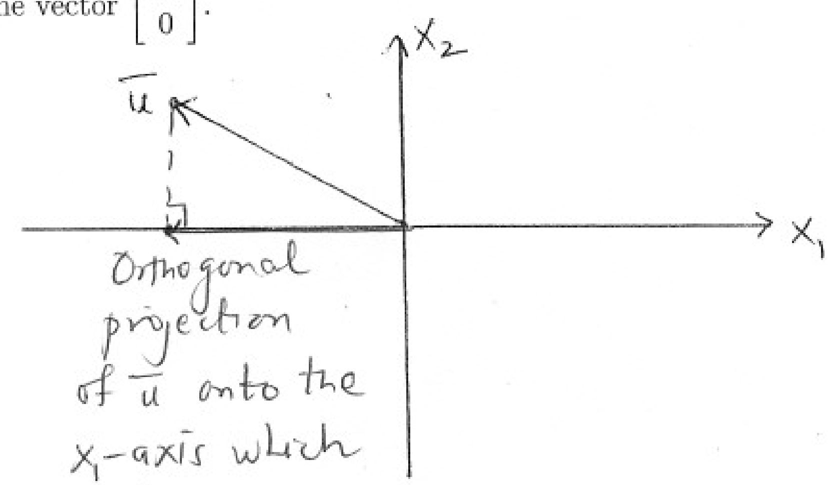
$$\det 3A = 3^4 \det A = 3^4 (20) = 1620.$$

(d) Let $B$ be a $5 \times 5$ matrix. The dimension of the eigenspace corresponding to the eigenvalue $-3$ of $B$ is 2. What is the dimension of Nul $(B + 3I)$? (Here $I$ is the $5 \times 5$ identity matrix.)

$\dim \text{Nul}(B + 3I) =$ dimension of eigenspace corresponding to eigenvalue $-3$

$= 2$.

(e) In the following figure, draw the orthogonal projection of the vector $\vec{u}$ onto the subspace spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.




Orthogonal projection of $\bar{u}$ onto the $x_1$-axis which is the subspace spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(f) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - 2x_4, 2x_3 + x_4).$$

What is the standard matrix of $T$?

$$\left[T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)\ T(\vec{e}_4)\right] = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(g) Let $T : M_{2\times 2} \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - c \\ b - d \end{bmatrix}.$$

i. Find $T\left(\begin{bmatrix} -2 & 5 \\ 0 & 20 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} -2 & 5 \\ 0 & 20 \end{bmatrix}\right) = \begin{bmatrix} -2-0 \\ 5-20 \end{bmatrix} = \begin{bmatrix} -2 \\ -15 \end{bmatrix}.$$

ii. Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find a matrix $A$ in $M_{2\times 2}$ such that $T(A) = \vec{b}$.

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

ie $\begin{bmatrix} a-c \\ b-d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\begin{matrix} a-c=1 \\ b-d=1 \end{matrix}$.

Choose $a=1,\ c=0,\ b=1,\ d=0$.

So one possible matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(There are infinitely many answers possible. Choose $a, b, c, d$ to satisfy the two equations.)