Math 205B Final Exam (100 points)

Name: ____

- Check that you have 8 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (15 points) Let W be the subspace spanned by the two vectors $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

   Let $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

   (a) Is the vector $\vec{y}$ in W? Explain.

   (b) Find a vector in W that is closest to $\vec{y}$.

   (c) Find a vector that is orthogonal to W.

2. (15 points) Orthogonally diagonalize the matrix $A = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$. (The eigenvalues of A are 25 and -25.)

3. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace.

(a) $W = \left\{ \begin{bmatrix} 2a - c + d \\ b - 2c - 2d \\ a + 3b + d \\ 2b + c + d \end{bmatrix} : a, b, c, d \text{ are real numbers.} \right\}.$

(b) All polynomials in $\mathbb{P}_3$ of the form $t + a$.

4. (12 points) Let $\vec{p_1}(t) = 2t - t^2$, $\vec{p_2}(t) = 2t$, $\vec{p_3}(t) = 2 - t$.

(a) Use coordinate vectors to show that $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for $\mathbb{P}_2$.

(b) Find the polynomial $\vec{q}$ in $\mathbb{P}_2$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

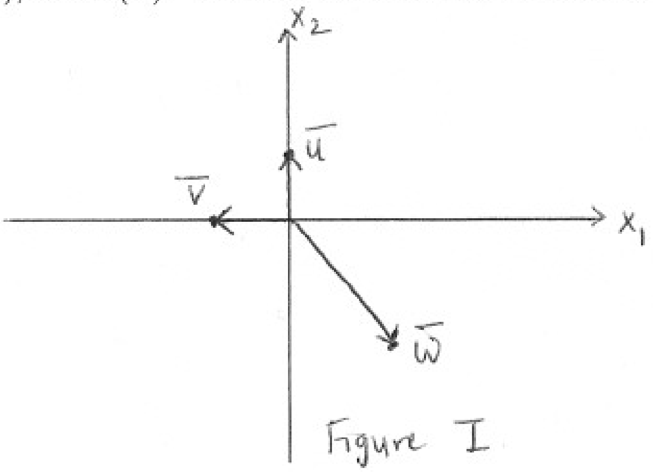
5. (10 points) Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^7$.

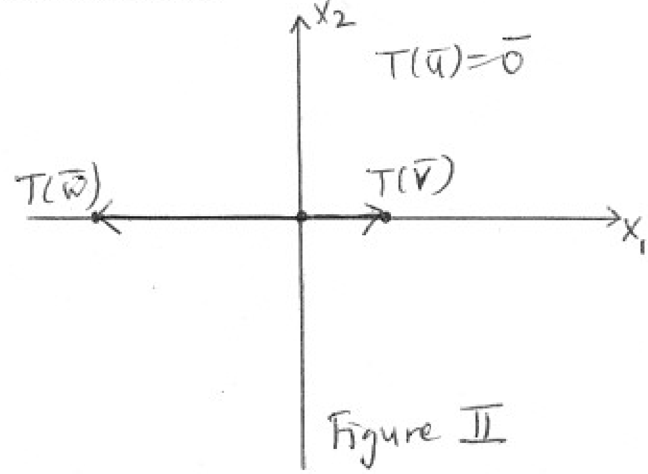
   (a) Show that $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$ is also a linearly independent set.

   (b) Is $\vec{v}_1$ in Span$\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$? Explain.

6. (8 points) Suppose U is an $n \times n$ orthogonal matrix. For every vector $\vec{x}$ in $\mathbb{R}^n$, show that the length of the vector $U\vec{x}$ is the same as the length of the vector $\vec{x}$. (Hint: Length of $U\vec{x}$ is $\sqrt{U\vec{x} \cdot U\vec{x}}$ and length of $\vec{x}$ is $\sqrt{\vec{x} \cdot \vec{x}}$. So it is enough to show that $U\vec{x} \cdot U\vec{x} = \vec{x} \cdot \vec{x}$.)

7. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ (A is a 2×2 matrix). Figure I below shows vectors $\vec{u}, \vec{v}$ and $\vec{w}$ and Figure II below shows vectors $T(\vec{u})$, $T(\vec{v})$, and $T(\vec{w})$. Use this information to answer the questions that follow.





(a) In Figure II, draw $T(\vec{v} + \vec{w})$.

(b) Which of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ (if any) are eigenvectors of A? What are the corresponding eigenvalues? Explain.

(c) Is T one-to-one? Explain.

8. (18 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

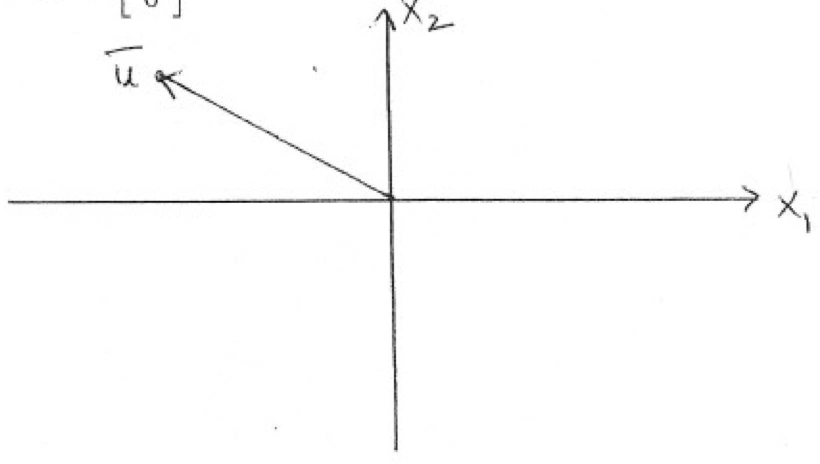
(a) Find the distance between the vector $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) A 2×4 matrix has rank 2. Find the dimension of the null space of this matrix.

(c) Suppose A is a 4×4 matrix with $\det A = 20$. What is $\det 3A$?

(d) Let B be a 5×5 matrix. The dimension of the eigenspace corresponding to the eigenvalue -3 of B is 2. What is the dimension of Nul $(B + 3I)$? (Here I is the 5×5 identity matrix.)

(e) In the following figure, draw the orthogonal projection of the vector $\vec{u}$ onto the subspace spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



(f) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - 2x_4, 2x_3 + x_4).$$

What is the standard matrix of T?

(g) Let $T : M_{2\times 2} \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - c \\ b - d \end{bmatrix}.$$

i. Find $T\left(\begin{bmatrix} -2 & 5 \\ 0 & 20 \end{bmatrix}\right)$.

ii. Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find a matrix A in $M_{2\times 2}$ such that $T(A) = \vec{b}$.