Final Exam, Math 205B (Linear Algebra)

This take-home exam is due by 5 PM on **Friday, December 12**. You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the appropriate place on the other side of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Matrix multiplications and reduced row echelon forms may be done on MATLAB or a calculator, but please show all other work.

1. (14 points) Find the complete solution of the system $\begin{pmatrix} 1 & 1 & 4 & 3 \\ 1 & 2 & 1 & 5 \\ 1 & 5 & -8 & 11 \\ -1 & 4 & -19 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \\ -6 \end{pmatrix}$.

2. (24 points) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 6 \\ 1 & 6 & 12 \end{pmatrix}$.

(a) Find the LU factorization of A. Since A is symmetric, you should also be able to find the factorization LDL^T of A.

(b) Find the determinants of L, D and L^T.

(c) Use your answer to (b) to find the determinant of A.

(d) Find L^{-1}, D^{-1} and $(L^T)^{-1}$.

(e) Use your answer to (d) to find A^{-1}.

3. (18 points) Find a basis for each of the four subspaces associated with the matrix

$$B = \begin{pmatrix} 1 & 2 & 5 & 8 \\ 3 & 2 & 7 & 4 \\ 9 & 2 & 13 & -8 \end{pmatrix}$$

What is the factored form of B that displays these bases?

4. (8 points) In the previous problem your column space basis should consist of two vectors in $\mathbb{R}^3$. What is their cross product? Is there anything interesting about the answer? Explain.

5. (14 points) Calculate the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 9 & 8 \\ 1 & 4 & 4 & 3 \\ 1 & 1 & 5 & 7 \end{vmatrix}$$

Are the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 9 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \\ 3 \\ 7 \end{pmatrix}$ linearly independent? How do you know?

6. (20 points) Explain how you can tell that R is a reflection matrix:

$$R = \frac{1}{3} \begin{pmatrix} -1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 & 0 & -1 \\ 2 & 0 & -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -1 & 0 & -2 \\ 2 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 2 \end{pmatrix}$$

What is the associated projection matrix? Find a basis for the subspace S of $\mathbb{R}^6$ that R reflects through, and a basis for $S^{\perp}$. Find also the eigenvalues and eigenvectors of R. (Hint: this last part may take some thought, but it should require little or no further calculation.)

7. (12 points) Recall Chris Richards's theorem that if C is a 2×2 matrix with distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and if

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad E = C - \Lambda$$

then the first column of E is a multiple of $\vec{v}_2$, and the second column of E is a multiple of $\vec{v}_1$. In the next problem I want you to show that the theorem holds even if the eigenvalues are not distinct. In this one let's do an example: find the eigenvalues and eigenvectors of

$$C = \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}.$$

(Don't be surprised if you can't find two independent eigenvectors.) What is the matrix E (defined above) in this example? Are its columns related to the eigenvectors? If so, how?

8. (20 points) Now let's try to prove the theorem mentioned in the previous problem. Suppose

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has a repeated eigenvalue λ.

(i) Show that λ must be $\frac{a+d}{2}$. (Hint: trace.)

(ii) Write down what the matrices Λ and E defined in the previous problem are in this case. Simplify E if you can. Since E is 2×2, the possible ranks for E are $0, 1, 2$. Explain why 2 is actually not possible.

(iii) Could the rank of E be 0? What would E be then? What would C be? What eigenvalues would C have? What are the eigenvectors? Are the columns of E multiples of these eigenvectors?

(iv) The interesting case is when the rank of E is 1 (as in the previous problem). What does this mean about the columns of E? Are they eigenvectors of C? Explain.

(v) What homework problem from this semester is part (iv) related to? Explain.

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________