# Math 206 — Final Exam

December 11, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover AND IS DOUBLE SIDED. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.
5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.
6. You may use any previously permitted calculator. However, you must state when you use it.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones and hats.
9. Remember that this is a chance to show what you've learned, and that the questions are just prompts.

| Problem | Points | Score |
|---|---|---|
| 1 | 09 | |
| 2 | 06 | |
| 3 | 13 | |
| 4 | 10 | |
| 5 | 12 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 06 | |
| 9 | 10 | |
| 10 | 07 | |
| 11 | 02 | |
| Total | 100 | |

Some formulae:

$$\nabla \times F = \left( \frac{dF_3}{dy} - \frac{dF_2}{dz}, \frac{dF_1}{dz} - \frac{dF_3}{dx}, \frac{dF_2}{dx} - \frac{dF_1}{dy} \right)$$

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

**1**. [9 points] Let $D$ be the region in the $xy$-plane which is bounded by $x^2 - y^2 = 9$, $x^2 - y^2 = 5$, $xy = 2$, and $xy = 1$. Compute

$$\int\int_D xy(x^2 + y^2)dxdy.$$

*Solution:* We let $u = x^2 - y^2$ and $v = xy$. The Jacobian of this is $(2x, -2y), (y, x)$ which has determinant $2x^2 + 2y^2$, and 1 over that is $1/2(x^2 + y^2)$. Then we have

$$\int_5^9 \int_1^2 v(x^2 + y^2)/2(x^2 + y^2)dvdu = \int_5^9 \int_1^2 v/2dvdu = 12.$$

**2**. [6 points] Fill in the bounds of the integral:

$$\int_0^1 \int_0^x f(x,y)dydx + \int_1^2 \int_0^1 f(x,y)dydx + \int_2^3 \int_{x-2}^1 f(x,y)dydx = \int_a^b \int_c^d f(x,y)dxdy.$$

$a =$

$b =$

$c =$

$d =$

*Solution:* If we draw the region, we see that $y$ can be at least 0 and at most 1, and $x$ goes from the line $x = y$ to the line $x = y + 2$ so our integral is

$$\int_0^1 \int_y^{y+2} f(x,y)dxdy.$$

**3**. [13 points] In this problem, let

$$F(x, y, z) = \left( e^{x^3}, \frac{3z}{y^2 + z^2}, -\frac{3y}{y^2 + z^2} \right),$$

and let $C$ be the curve parametrized by $p(t) = (\cos(t), \sin(2t), \cos(2t))$ with $0 \leq t \leq 2\pi$.

**a**. [5 points] Set up (but do not compute) the arc length integral for the curve $C$?

*Solution:* The formula for arclength is to integrate $|p'(t)|$ from 0 to $2\pi$. This is

$$\int_0^{2\pi} \sqrt{\sin^2(t) + 4}dt.$$

**b**. [8 points] Compute

$$\int_C F \cdot dL.$$

*Solution:* We need to compute $\int_0^{2\pi} F(p(t)) \cdot p'(t)dt$. We have that

$$F(p(t)) = \left( e^{\cos^3(t)}, \frac{3\cos(2t)}{\sin^2(2t) + \cos^2(2t)}, -\frac{3\sin(2t)}{\sin^2(2t) + \cos^2(2t)} \right).$$

We have that $p'(t) = (-\sin(t), 2\cos(2t), -2\sin(2t))$. Combining all this we get

$$\int_0^{2\pi} - e^{\cos^3(t)} \sin(t) + 6dt = 12\pi,$$

after the $u$-sub $u = \cos(x)$ and $du = -\sin(x)dx$.

**4**. [10 points] Let $f(x)$ and $g(y)$ be functions of one variable with continuous second derivatives. Let $u(x, y) = f(x + g(y))$. Compute

$$u_x u_{xy} - u_y u_{xx} = \frac{du}{dx}\frac{d^2u}{dxdy} - \frac{du}{dy}\frac{d^2u}{dx^2}.$$

*Solution:* This is a chain rule problem. First let's calculate $u_x$ and $u_y$. We have that $\text{Jac}u = \text{Jac}f * \text{Jac}h$ where $h(x, y) = x + g(y)$. So:

$$\begin{aligned} \text{Jac}(h) &= (1, g'(y)) \\ \text{Jac}(f)|_{h(x,y)} &= f'(x + g(y)) \\ \text{Jac}(u) &= (f'(x + g(y)), f'(x + g(y))g'(y)). \end{aligned}$$

So $u_x = f'(x+g(y))$ and $u_y = f'(x+g(y))g'(y)$. To compute $u_{xy}$ we take the derivative of $u_y$ with respect to $x$. This gives us (after a similar chain rule calculation) $u_{xy} = f''(x+g(y))g'(y)$. Finally, we get $u_{xx}$ by differentiating $u_x$ with respect to $x$ and also get $u_{xx} = f''(x + g(y))$. Putting all this together gives:

$$u_x u_{xy} - u_y u_{xx} = f'(x + g(y))f''(x + g(y))g'(y) - f'(x + g(y))g'(y)f''(x + g(y)) = 0.$$

**5**. [12 points] Given a sheet of metal lying in the $xy$-plane with corners at $(7, 0), (6, 0), (3, -4), (5, -8)$, and $(7, -3)$ and mass density $x$ kg/m. What is the total mass of this irregularly shaped metal?

*Solution:* We can either to the integral of $x$ over this messy region. OR we can do the integral of $F(x,y) = (F_1, F_2)$ with $\frac{dF_2}{dx} - \frac{dF_1}{dy} = x$ because we can then use Green's theorem and integrate around the boundary of $F$. Our boundary can be parametrized as several line segments $C_1, C_2, C_3, C_4, C_5$ with the following parametrizations:

$$\begin{aligned} C_1\colon p(t) &= (7,0) + t(-1,0) \\ C_2\colon p(t) &= (6,0) + t(-3,-4) \\ C_3\colon p(t) &= (3,-4) + t(2,-4) \\ C_4\colon p(t) &= (5,-8) + t(2,5) \\ C_5\colon p(t) &= (7,-3) + t(0,3) \end{aligned}$$

With all the parametrizations having $0 \le t \le 1$. Then we just need to integrate $F(p(t)) \cdot p'(t)$ over each curve. Note that $p'(t)$ is just the second vector for each of those parametrizations. We can choose $F$ to be $F(x,y) = (-xy, 0)$. We then get the following 5 integrals to compute:

$$\begin{aligned} \int_0^1 0 dt &= 0 \\ \int_0^1 -12t(6-3t) dt &= -24 \\ \int_0^1 -2(3+2t)(-4-4t) dt &= \frac{148}{3} \\ \int_0^1 -2(5+2t)(-8+5t) dt &= \frac{193}{3} \\ \int_0^1 0 dt &= 0. \end{aligned}$$

So the sum of all these integrals is $\frac{269}{3}$.

**6**. [15 points] Let $S$ be the surface parametrized by $p(s,t) = (4s\cos(t), 4s\sin(t), 4s^2)$, with $0 \le s \le 1$ and $0 \le t \le 2\pi$, and let $C$ be the boundary of $S$. Also let $F(x,y,z) = (yz\,\mathrm{e}^{xyz}, xz\,\mathrm{e}^{xyz} - z, xy\,\mathrm{e}^{xyz} - y + x)$.

**a**. [5 points] What is $\nabla \times F$? Is $F$ a conservative vector field?

*Solution:* The curl of $F$ is $(0,-1,0)$ so $F$ is NOT conservative.

**b**. [5 points] What is the parametrization of $C$?

*Solution:* Since $C$ is the boundary of $S$ we can parametrize it by letting $s = 1$ in the parametrization of $S$. This gives $C = f(t) = (4\cos(t), 4\sin(t), 4)$ which happens to be the circle of radius 4 parallel to the $xy$-plane at height $z = 4$.

**c**. [5 points] Use Stokes Theorem to compute $\int\int_S \nabla \times F dA$ by finding another surface that also has $C$ as its boundary.

*Solution:* This is easier if we take the disc that is the fill in of that circle, $D$. This is parametrized by $q(s,t) = (4s\cos(t), 4s\sin(t), 4)$ and has $q_s \times q_t = (0,0,4)$. From Stokes Theorem (since our new surface, $D$, and $S$ share a boundary) we have that

$$\int\int_S \nabla F dA = \int\int_D \nabla F dA = \int_0^1 \int_0^{2\pi} (0,-1,0)\cdot(0,0,4) dt ds = 0.$$

**7.** [10 points] True or False (no partial credit)

**a.** [2 points] It is true for any $a, b, c \in \mathbb{R}^3$ we have that $((c \times a) \cdot c) + ((c \times b) \cdot c) = 0$?

*Solution:* TRUE

**b.** [2 points] $x^2 + y^2 - 2y = z$ is the equation of a cone.

*Solution:* FALSE

**c.** [2 points] $p(t) = (t - 1, 2t - 3, 5t)$ is the equation of a line through the point $(1, 1, 15)$.

*Solution:* FALSE

**d.** [2 points] $\tan(\theta) = -1$ in polar coordinates defines the same curve as $x + y = 0$ in rectilinear coordinates.

*Solution:* TRUE

**e.** [2 points] if $f(x, y) = xy$ and $C\colon p(t) = (t, 1 + t^2)$ for $0 \le t \le 2$ then $\int_C \nabla f \, dL = 10$.

*Solution:* TRUE

**8**. [6 points] Let $f(x, y)$ be a function of two variables.

**a**. [3 points] State the limit definition of $\frac{df}{dx}(0, 1)$.

*Solution:*

$$\lim_{h\to 0} \frac{f(h, 1) - f(0, 1)}{h}.$$

**b**. [3 points] Let $n = (1, 2)$. State the limit definition of $\frac{df}{dn}(0, 1)$.

*Solution:*

$$\lim_{h\to 0} \frac{f(h, 1 + 2h) - f(0, 1)}{\sqrt{5}h}.$$

**9**. [10 points] Let $f(x, y) = (\sqrt{xy}, \sqrt{x + y})$.

**a**. [5 points] What is the equation for the tangent plane at $(2, 2)$?

*Solution:* We need the Jacobian of $f$ at $(2, 2)$ This is

$$\begin{pmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2\sqrt{x+y}} & \frac{1}{2\sqrt{x+y}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Using the formula

$$f(a, b) + \text{Jac}(f)(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix} = z$$

we get

$$(2, 2) + ((x - 2)/2 + (y - 2)/2, (x - 2)/4 + (y - 2)/4).$$

**b**. [5 points] Use part (a) to approximate $f(1, 2)$.

*Solution:* Plugging in $(1, 2)$ to the solution we have, we get $(2, 2) + (-1/2, -1/4) = (1.5, 1.75) \sim (\sqrt{2}, \sqrt{3})$.

**10**. [7 points] Find the critical points and classify them (local min, local max, saddle point) of the function $f(x,y) = x + y - \ln(xy)$.

*Solution:* We find the gradient of $f(x,y) = \left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right)$. and set it equal to $(0,0)$. This only happens at $x = y = 1$. To figure out what type of critical point this is we compute the Hessian

$$\begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

Since the determinants are both positive, this is a local minimum.

**11**. [2 points] What was your favorite topic this semester?

*Solution:* It's so hard to choose!