

Math 106: Review for Final Exam, Part II - SOLUTIONS

1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.

We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.

$$\begin{aligned} f(x) &= x^{1/3} & f(27) &= 3 \\ f'(x) &= \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} & f'(27) &= \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27} \\ f''(x) &= -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}} & f''(27) &= -\frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187} \end{aligned}$$

Now plug in to the Taylor polynomial formula with $x_0 = 27$.

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2$$

Finally, evaluate at $x = 28$.

$$\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797$$

2. What is the largest possible error that could have occurred in your previous estimate?

We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

$$K_3 = \max \text{ of } |f'''(x)| \text{ on } [27, 28] = \max \text{ of } \left| \frac{10}{27x^{8/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}$$

Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{10}{3!} \frac{1}{177147} |28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$.

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

(a) $\int_1^{\infty} \frac{7 + 5 \sin x}{x^2} dx$

For all $x \geq 1$, we have $0 \leq \frac{7 + 5 \sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = 12 \frac{1}{x^2}$ because the maximum of $\sin x$ is 1.

$$\begin{aligned} 12 \int_1^{\infty} \frac{dx}{x^2} &= 12 \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} \\ &= 12 \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_1^t \\ &= 12 \lim_{t \rightarrow \infty} \left[\frac{-1}{t} - \frac{-1}{1} \right] \\ &= 12[0 - (-1)] \\ &= 12 \end{aligned}$$

Therefore, the original integral in question must converge to a value less than 12.

(b) $\int_1^{\infty} \frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}} dx$

For $x \geq 1$, we have $\frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt[3]{10x^{12} + 17x^{12}}} \geq 0$. (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)

But $\frac{2x^3}{\sqrt[3]{10x^{12} + 17x^{12}}} = \frac{2x^3}{\sqrt[3]{27x^{12}}} = \frac{2x^3}{3x^4} = \frac{2}{3} \frac{1}{x}$ and we know that $\frac{2}{3} \int_1^\infty \frac{dx}{x}$ diverges (compute for yourself or notice that $p = 1$).

Therefore the original integral must also diverge.

4. The probability density function (pdf) of the weights of newborn toads in a certain pond is given by $f(x) = \frac{k}{(x+1)^4}$, where x is the weight (in ounces). Note that the domain is $x \geq 0$ since no toad can have a negative weight.

- (a) What must be the value of k ?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$\begin{aligned} \int_0^\infty \frac{k}{(x+1)^4} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{k}{(x+1)^4} dx \\ &= \lim_{t \rightarrow \infty} \left. \frac{k(x+1)^{-3}}{-3} \right|_0^t \\ &= \lim_{t \rightarrow \infty} \left. \frac{k}{-3(x+1)^3} \right|_0^t \\ &= \lim_{t \rightarrow \infty} \frac{k}{-3(t+1)^3} - \frac{k}{-3(0+1)^3} \\ &= 0 - \frac{k}{-3} \\ &= \frac{k}{3} \end{aligned}$$

So, we have $k/3 = 1$ or $k = 3$.

- (b) What fraction of the newborn toads weigh more than one ounce?

$$\begin{aligned} \int_1^\infty \frac{3}{(x+1)^4} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{3}{(x+1)^4} dx \\ &= \lim_{t \rightarrow \infty} \left. \frac{3}{-3(x+1)^3} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \\ &= 0 - \frac{1}{-8} \\ &= \frac{1}{8} \end{aligned}$$

Note that we could instead have computed $1 - \int_0^1 \frac{3}{(x+1)^4} dx$ and gotten the same answer.

5. Decide if each of the following sequences $\{a_k\}_{k=1}^\infty$ converges or diverges. If a sequence converges, compute its limit.

- (a) $a_k = 3 + \frac{1}{10^k}$ Terms are 3.1, 3.01, 3.001, 3.0001, ...

$\lim_{k \rightarrow \infty} \left(3 + \frac{1}{10^k} \right) = 3$, so the sequence converges to 3.

- (b) $a_k = (-1)^k$ Terms are $-1, 1, -1, 1, \dots$

$\lim_{k \rightarrow \infty} (-1)^k$ doesn't exist, so the sequence diverges.

(c) $a_k = \frac{3 + 5k}{7 + 2k}$ Terms are 8/9, 13/11, 18/13, 23/15, ...
 $\lim_{k \rightarrow \infty} \frac{3 + 5k}{7 + 2k} = \frac{5}{2}$ (by L'Hopital's Rule or by inspection), so the sequence converges to $\frac{5}{2}$.

6. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) $3.1 + 3.01 + 3.001 + 3.0001 + \dots$
 $\lim_{k \rightarrow \infty} a_k = 3 \neq 0$, so the series diverges by the nth Term Test. (We keep adding 3's forever.)
 [Compare this with the first sequence of the previous problem.]

(b) $1 + 1/2 + 1/3 + 1/4 + \dots$
 This is the famous Harmonic Series, which diverges *even though* the terms do approach 0. We can use the Integral Test: $\int_1^{\infty} \frac{1}{x} dx$ diverges, which means that $\sum_{k=1}^{\infty} \frac{1}{k}$ must diverge too.

(c) $5 - 5/3 + 5/9 - 5/27 + \dots$
 This is a geometric series with $r = -\frac{1}{3}$, so it converges to $\frac{a}{1 - r} = \frac{5}{1 - (-1/3)} = \frac{15}{4}$.

7. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k+1}}$ [Alternating Series Test]

The terms of this series alternate in sign.

And, $\frac{1}{\sqrt[3]{2}} \geq \frac{1}{\sqrt[3]{3}} \geq \frac{1}{\sqrt[3]{4}} \geq \dots \geq 0$.

And, $\lim_{k \rightarrow \infty} \frac{1}{\sqrt[3]{k+1}} = 0$.

Therefore, by the Alternating Series Test, the series must converge.

We know that any two consecutive partial sums will provide upper and lower bounds:

lower bound = $S_1 = \frac{-1}{\sqrt[3]{2}}$ upper bound = $S_2 = \frac{-1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}}$

[To get better bounds, use later partial sums, such as S_5 and S_6 .]

(b) $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k (k!)^2}$ [Ratio Test]

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \frac{[2(k+1)]!}{3^{k+1} [(k+1)!]^2} \cdot \frac{(2k)!}{3^k (k!)^2} \\ &= \lim_{k \rightarrow \infty} \frac{(2k+2)!}{(2k)!} \cdot \frac{3^k}{3^{k+1}} \cdot \frac{(k!)^2}{[(k+1)!]^2} \\ &= \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1)}{1} \cdot \frac{1}{3} \cdot \frac{1}{(k+1)^2} \\ &= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3} \\ &= \frac{4}{3} \end{aligned}$$

Use L'Hopital or divide each term by k^2 .

Since the limit of the ratio is greater than 1, the series diverges.

(c) $\sum_{k=1}^{\infty} \left(\frac{1}{100} + \frac{1}{k^5} \right)$ [nth Term Test]

$\lim_{k \rightarrow \infty} \left(\frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0$, so, by the nth Term Test, the series diverges.

(d) $\sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3}$ [Comparison Test]

For $k \geq 1$, we have $\frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} > \frac{\sqrt{9k^8}}{12k^5 + 3k^5} > 0$.

But, $\frac{\sqrt{9k^8}}{12k^5 + 3k^5} = \frac{3k^4}{15k^5} = \frac{1}{5} \frac{1}{k}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (use Integral Test or note that $p = 1$).

Therefore, the original series, which has larger terms, must diverge also.

(e) $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$ [Integral Test]

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx \\ &= \lim_{t \rightarrow \infty} \int_{x=2}^{x=t} \frac{1}{u^2} du && \text{Substitute } u = \ln x, \text{ so } du = \frac{dx}{x}. \\ &= \lim_{t \rightarrow \infty} \left. \frac{u^{-1}}{-1} \right|_{x=2}^{x=t} \\ &= \lim_{t \rightarrow \infty} \left. \frac{-1}{\ln x} \right|_2^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{-1}{\ln t} - \frac{-1}{\ln 2} \right] \\ &= 0 - \frac{-1}{\ln 2} \\ &= \frac{1}{\ln 2} \end{aligned}$$

The integral converges, so the series must converge too.

Further, we know that $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx \leq \sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2} \leq a_2 + \int_2^{\infty} \frac{1}{x(\ln x)^2} dx$.

Therefore, our lower bound is $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}$.

And our upper bound is $a_2 + \int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}$.

8. Does the first series from the previous problem converge absolutely or conditionally?

Recall that a series $\sum_{k=1}^{\infty} \mathbf{a}_k$ converges *absolutely* if $\sum_{k=1}^{\infty} |\mathbf{a}_k|$ converges; a series $\sum_{k=1}^{\infty} \mathbf{a}_k$ converges

conditionally if $\sum_{k=1}^{\infty} \mathbf{a}_k$ converges but $\sum_{k=1}^{\infty} |\mathbf{a}_k|$ diverges.

$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{\sqrt[3]{k+1}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+1}}$, which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges conditionally.

9. Compute the radius and interval (including endpoints) of convergence for $\sum_{k=1}^{\infty} \frac{(x+3)^k}{k \cdot 5^k}$.

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{\frac{(x+3)^{k+1}}{(k+1) \cdot 5^{k+1}}}{\frac{(x+3)^k}{k \cdot 5^k}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(x+3)^{k+1}}{(x+3)^k} \frac{k}{k+1} \frac{5^k}{5^{k+1}} \right| && \text{Use L'Hopital on the middle fraction.} \\ &= \left| (x+3) \cdot 1 \cdot \frac{1}{5} \right| \\ &= \left| \frac{x+3}{5} \right| \end{aligned}$$

So, we are guaranteed convergence when $\left| \frac{x+3}{5} \right| < 1$. But this is equivalent to the following.

$$\begin{aligned} -1 &< \frac{x+3}{5} < 1 \\ -5 &< x+3 < 5 \\ -8 &< x < 2 \end{aligned}$$

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At $x = 2$, we have $\sum_{k=1}^{\infty} \frac{(2+3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{1}{k}$, which is the Harmonic Series and thus diverges.

At $x = -8$, we have $\sum_{k=1}^{\infty} \frac{(-8+3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which converges by the Alternating Series Test.

Thus, the interval of convergence is $-8 \leq x < 2$ and the radius of convergence is 5.

10. Find the complete Taylor series (in summation notation) for $f(x) = \ln(1-x)$ about $x = 0$.

$$\begin{aligned} f(x) &= \ln(1-x) & f(0) &= 0 \\ f'(x) &= \frac{-1}{1-x} & f'(0) &= -1 \\ f''(x) &= \frac{-1}{(1-x)^2} & f''(0) &= -2 \\ f'''(x) &= \frac{-2}{(1-x)^3} & f'''(0) &= -6 \\ f^{(4)}(x) &= \frac{-6}{(1-x)^4} & f^{(4)}(0) &= -24 \end{aligned}$$

Now plug in to the Taylor series formula with $x_0 = 0$.

$$\begin{aligned} f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots &= 0 - 1(x) + \frac{-1}{2}x^2 + \frac{-2}{6}x^3 + \dots \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \\ &= \sum_{k=1}^{\infty} \frac{-x^k}{k} \end{aligned}$$

11. Evaluate the following exactly.

(a) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \dots$

We recall that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

The series in question is the series for e^x with $x = -1$; therefore, it converges to e^{-1} .

(b) $\frac{8}{3} - \frac{8}{9} + \frac{8}{27} - \frac{8}{81} + \dots$

We recognize this as a geometric series with $r = -1/3$ and $a = 8/3$; therefore, it converges to

$$\frac{a}{1-r} = \frac{8/3}{1 - (-1/3)} = 2.$$

(c) $1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \dots$

We recall that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

The series in question is the series for $\cos x$ with $x = \pi$; therefore, it converges to $\cos \pi$, which is -1 .

12. (a) **Write the complete series equal to $\int_0^1 e^{-x^2} dx$ and show that it converges.**

We know $e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \dots$, so by substitution we obtain the following.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$\begin{aligned} \text{Thus, } \int_0^1 e^{-x^2} dx &= \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right] dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right]_0^1 \\ &= 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} \end{aligned}$$

The terms of this series alternate in sign.

$$\text{And, } 1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1^7}{7 \cdot 3!} \geq \dots \geq 0.$$

$$\text{And, } \lim_{k \rightarrow \infty} \frac{1}{(2k+1)k!} = 0.$$

Therefore, by the Alternating Series Test, the series must converge.

- (b) **If $f(x) = e^{-x^2}$, what is $f^{(400)}(0)$? What is $f^{(401)}(0)$?**

In the previous part, we found the following.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

We know that $f^{(400)}(0)$ will appear in the Taylor series for $f(x)$ in the coefficient of the x^{400} term, which is $\frac{f^{(400)}(0)}{400!} x^{400}$.

From above, we see that term is $\frac{x^{400}}{200!}$.

Setting them equal gives $\frac{f^{(400)}(0)}{400!} x^{400} = \frac{x^{400}}{200!}$, which means that $f^{(400)}(0) = \frac{400!}{200!}$.

[Note that this does *not* simplify to $200!$. Rather, $\frac{400!}{200!} = 400 \cdot 399 \cdot \dots \cdot 202 \cdot 201.$]

We know that $f^{(401)}(0)$ will appear in the Taylor series for $f(x)$ in the coefficient of the x^{401} term, which is $\frac{f^{(401)}(0)}{401!}x^{401}$.

However, the Taylor series above for $f(x)$ has no terms with odd powers of x , meaning that the coefficients of all those terms must be 0.

Therefore, we know that $f^{(401)}(0) = 0$.