Math 106: Review for Final Exam, Part II

1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.

2. What is the largest possible error that could have occurred in your previous estimate?

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

   (a) $\displaystyle\int_1^\infty \frac{7+5\sin x}{x^2}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{1+3x^2+2x^3}{\sqrt[3]{10x^{12}+17x^{10}}}\,dx$

4. The probability density function (pdf) of the weights of newborn toads in a certain pond is given by $f(x) = \dfrac{k}{(x+1)^4}$, where x is the weight (in ounces). Note that the domain is $x \geq 0$ since no toad can have a negative weight.

   (a) What must be the value of k?

   (b) What fraction of the newborn toads weigh more than one ounce?

5. Decide if each of the following sequences $\{a_k\}_{k=1}^{\infty}$ converges or diverges. If a sequence converges, compute its limit.

   (a) $a_k = 3 + \dfrac{1}{10^k}$

   (b) $a_k = (-1)^k$

   (c) $a_k = \dfrac{3 + 5k}{7 + 2k}$

Strategy. The following is a good order in which to consider the various series convergence tests.

(a) Do the individual terms approach 0? If not, the nth Term Test tells you the series must diverge.

(b) Is the series geometric? (That is, do you multiply by the same constant r to get from each term to the next?) If so, the series converges if $|r| < 1$ and diverges otherwise.

(c) Does the series contain something such as $(-1)^k$ or $(-1)^{k+1}$ or $\cos(k\pi)$ that makes its terms alternate? If so, try the Alternating Series Test.

(d) Does the series contain a factorial ($k!$) or exponential (such as 2^k or e^k)? If so, try the Ratio Test.

(e) If the series has positive terms, does it remind you of a simpler series (especially a p-series: powers of k such as $1/k$ or $1/k^2$)? If so, try the Comparison Test.

(f) Is the formula something you can integrate easily? If so, try the Integral Test.

6. Decide if each of the following series converges or diverges. If a series converges, find its value.

   (a) $3.1 + 3.01 + 3.001 + 3.0001 + ...$

   (b) $1 + 1/2 + 1/3 + 1/4 + ...$

   (c) $5 - 5/3 + 5/9 - 5/27 + ...$

7. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k+1}}$

   (b) $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k (k!)^2}$

   (c) $\sum_{k=1}^{\infty} (\frac{1}{100} + \frac{1}{k^5})$

   (d) $\sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3}$

(e) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$

8. Does the first series from the previous problem converge absolutely or conditionally?

   Recall that a series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges *absolutely* if $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges;

   a series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges *conditionally* if $\displaystyle\sum_{k=1}^{\infty} a_k$ converges but $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges.

9. Compute the radius and interval (including endpoints) of convergence for $\displaystyle\sum_{k=1}^{\infty} \frac{(x+3)^k}{k \cdot 5^k}$.

10. Find the complete Taylor series (in summation notation) for $f(x) = \ln(1-x)$ about $x = 0$.

11. Evaluate the following exactly.

(a) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots$

(b) $\frac{8}{3} - \frac{8}{9} + \frac{8}{27} - \frac{8}{81} + \ldots$

(c) $1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \ldots$

12. (a) Write the complete series equal to $\int_0^1 e^{-x^2}\, dx$ and show that it converges.

(b) If $f(x) = e^{-x^2}$, what is $f^{(400)}(0)$? What is $f^{(401)}(0)$?