Math 105: Review for Final Exam, Part II - SOLUTIONS

1. **Consider the function $f(x) = x^3 \ln x$ on the interval $[1/e, e^2]$.**

   (a) **Find the x- and y-coordinates of any and all local extrema and classify each as a local maximum or local minimum.**

$$f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x}$$
$$0 = x^2(3\ln x + 1)$$
$$\Rightarrow x^2 = 0 \text{ (never on our domain) or } \ln x = -1/3, \text{ which means } x = e^{-1/3}$$

	$0 < x < e^{-1/3}$	$e^{-1/3} < x$
f'	negative	positive
f	$\searrow$	$\nearrow$

y-value: $f(e^{-1/3}) = (e^{-1/3})^3 \ln(e^{-1/3}) = (e^{-1})(-1/3) = (-1/3e)$
So, f has a local minimum at $(e^{-1/3}, -1/3e)$.

   (b) **Find the x- and y-coordinates of any and all global extrema and classify each as a global maximum or global minimum.**

We check the y-values at the local extrema and the endpoints.
$f(1/e) = (1/e)^3 \ln(1/e) = (1/e^3)(-1) = (-1/e^3)$
$f(e^{-1/3}) = (-1/3e)$ from above
$f(e^2) = (e^2)^3 \ln(e^2) = (e^6)(2) = 2e^6$
So, f has a global minimum at $(e^{-1/3}, -1/3)$ and a global maximum at $(e^2, 2e^6)$.

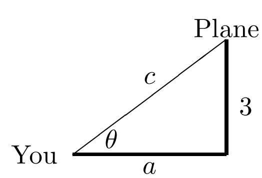
   (c) **Find the x-coordinate(s) of any and all inflection points.**

$$f''(x) = 2x(3\ln x + 1) + x^2(3 \cdot \frac{1}{x} + 0)$$
$$0 = 6x \ln x + 2x + 3x$$
$$0 = x(6\ln x + 5)$$
$$\Rightarrow x = 0 \text{ (never on our domain) or } \ln x = -5/6, \text{ which means } x = e^{-5/6}$$

	$0 < x < e^{-5/6}$	$e^{-5/6} < x$
f''	negative	positive
f	concave down	concave up

So, the x-value of the inflection point of f is $x = e^{-5/6}$.

2. **You are watching a plane flying toward your position at a constant height of 3 miles and a speed of 500 miles per hour relative to the ground. At the moment when the plane is 5 miles from you (diagonally), at what rate is the angle of your vision toward the plane changing?**



We know $\frac{da}{dt}$, and we want to find $\frac{d\theta}{dt}$.

So, we write an equation that relates a and θ and then differentiate implicitly with respect to time t.

$$\begin{aligned}
\tan\theta &= \frac{3}{a} \\
\sec^2\theta \frac{d\theta}{dt} &= -\frac{3}{a^2}\frac{da}{dt} \\
\frac{d\theta}{dt} &= -\frac{3}{a^2}\frac{da}{dt}\cos^2\theta
\end{aligned}$$

At the moment in question, $c = 5$, so by the Pythagorean Theorem we know that $a = 4$ and that $\cos\theta = \frac{4}{5}$. Finally, we are told that $\frac{da}{dt} = -500$.

So, $\frac{d\theta}{dt} = -\frac{3}{4^2}(-500)\left(\frac{4}{5}\right)^2 = 60$ radians per hour.

3. **Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is \$9.00 per container, what dimensions will give the largest volume?**

area of circle $= \pi r^2$ **lateral area of cylinder $= 2\pi rh$** **volume of cylinder $= \pi r^2 h$**

Objective function: volume $= V = \pi r^2 h$

We need to get this down to a function of just one variable, so we use the

$$\begin{aligned}
\textit{constraint equation}: \text{cost} = 900 &= 3\cdot 2\cdot \pi r^2 + 5\cdot 2\pi rh \\
900 &= 6\pi r^2 + 10\pi rh \\
900 - 6\pi r^2 &= 10\pi rh \\
\frac{900 - 6\pi r^2}{10\pi r} &= h
\end{aligned}$$

Substituting this back into the objective function gives

$V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r\cdot\frac{900 - 6\pi r^2}{10} = \frac{1}{10}(900r - 6\pi r^3)$.

Now that we have V as a function of just one variable, we find its maximum.

$$\begin{aligned}
V'(x) &= \frac{1}{10}(900 - 18\pi r^2) \\
0 &= \frac{1}{10}(900 - 18\pi r^2) \\
&\Rightarrow 18\pi r^2 = 900 \\
&\Rightarrow r^2 = \frac{50}{\pi} \\
&\Rightarrow r = \sqrt{\frac{50}{\pi}}
\end{aligned}$$

	$0 < x < \sqrt{50/\pi}$	$\sqrt{50/\pi} < x$
f'	positive	negative
f	↗	↘

Thus, we have in fact found the global maximum at $r = \sqrt{50/\pi}$.

And $h = \frac{900 - 6\pi r^2}{10\pi r} = $ ...much simplifying... $= \sqrt{\frac{72}{\pi}} \approx 4.787$ inches.

4. **State the Intermediate Value Theorem and use it to show that $f(x) = x^3 - x + 1$ has a root in $[-2, 0]$.**

   IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

   For the function given above, $f(-2) = -5$ and $f(0) = 1$. Since 0 is a number between -5 and 1, the IVT says there is a number c between -2 and 0 such that $f(c) = 0$; this c is the desired root.

5. **Use Newton's Method to find a root of $f(x) = x^3 - x + 1$ correct to three decimal places.**

   Recall that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - x_n + 1}{3x_n^2 - 1}$.

   Using a calculator with initial guess $x_0 = -1$, we get the following.

$$\begin{aligned} x_0 &= -1.5 \\ x_1 &= -1.34782... \\ x_2 &= -1.32520... \\ x_3 &= -1.32471... \\ x_4 &= -1.32471... \\ x_5 &= -1.32471... \end{aligned}$$

   So, the root is at approximately $x = -1.324$ (truncated) or $x = -1.325$ (rounded).

6. **What (if anything) does the Extreme Value Theorem say about $f(x) = x^2$ on each of the following intervals?**

   EVT: If f is continuous on $[a, b]$, then f has both a maximum and a minimum on $[a, b]$.

   (a) $\mathbf{[1, 4]}$

   f has a maximum and a minimum on $[1, 4]$

   (b) $\mathbf{(1, 4)}$

   The EVT doesn't apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

7. **State the Mean Value Theorem and find the value of the constant c that the theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.**

   MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c between a and b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

   For our function, we have $\dfrac{f(3) - f(0)}{3 - 0} = \dfrac{30 - 0}{3} = 10$.

   And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

   So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$.

8. **Find the following.**

   (a) $\displaystyle\int_1^7 \frac{3}{x}\,dx = 3\ln|x|\Big|_1^7 = 3\ln 7 - 3\ln 1 = 3\ln 7$

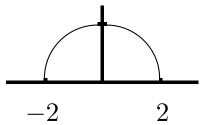
   (b) $\displaystyle\int_1^4 (1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5})\,dx = \left[x + x^2 + \frac{x^4}{4} + 4\frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4}\right]_1^4$

   $$= \left(4 + 4^2 + \frac{4^4}{4} + 4\frac{4^{3/2}}{3/2} + \frac{4^{-4}}{-4}\right) - \left(1 + 1^2 + \frac{1^4}{4} + 4\frac{1^{3/2}}{3/2} + \frac{1^{-4}}{-4}\right) = \frac{309245}{3072} \approx 100.66569$$

   Note that we have used the facts that $\sqrt{x} = x^{1/2}$ and $1/x^5 = x^{-5}$.

(c) $\displaystyle\int_0^2 e^{3x}\,dx = \frac{e^{3x}}{3}\Big|_0^2 = \frac{e^6}{3} - \frac{e^0}{3} = \frac{e^6-1}{3}$

(d) $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$ This integral represents the area of a semicircle of radius 2.



(e) $\displaystyle\frac{d}{dx}\int_1^x \sin\sqrt{t}\,dt = \sin\sqrt{x}$ The derivative of the area function is the original function.

(f) $\displaystyle\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)^2$

This represents the limit of a right-hand sum as the number (n) of rectangles goes to infinity.

As k goes from 1 to n, the expression $\left(1+\frac{2k}{n}\right)$ takes on the values $\left(1+\frac{2}{n}\right)$, $\left(1+\frac{4}{n}\right)$, …, $\left(1+\frac{2n}{n}\right)$; the first of these values is just to the left of 1 and the last is equal to 3, so we see that we are looking at the function on the interval $[1,3]$.

The $\frac{2}{n}$ out front is our Δx, which confirms that we are dealing with an interval of length 2 being subdivided into n equal subintervals.

Finally, each of the x-values is being squared, so the function in question must be $f(x) = x^2$.

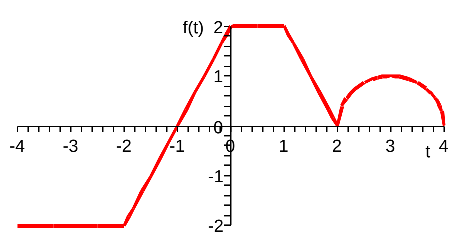
Thus, we see that the expression is the area under $f(x) = x^2$ on $[1,3]$.
Its value is $\displaystyle\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}$.

9. **Water is leaking out of a tank at a decreasing rate. Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.**

time (min)	0	2	4	6	8
rate (gal/min)	15	11	8	4	3

$$\text{overestimate} = L_4 = (15+11+8+4)(2) = 76$$
$$\text{underestimate} = R_4 = (11+8+4+3)(2) = 52$$

10. **Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.**



Let $G(x) = \int_0^x f(t)\ dt$ and $H(x) = \int_{-3}^x f(t)\ dt$.

(a) **Compute $G(2)$, $G(4)$, and $H(4)$.**

$G(2)$ is the area under f between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$.

Similarly, $G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 3 + \frac{\pi}{2}$.

$H(4)$ is the area under f between $t = -3$ and $t = 4$. Remember that area below the t-axis counts as negative.

$$\begin{aligned} H(4) &= -\left(2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1\right) + \frac{1}{2} \cdot 2 \cdot 1 + [\text{area under } f \text{ from 0 to 4, found above as } G(4)] \\ &= -2 + \left[3 + \frac{\pi}{2}\right] \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

(b) **Where is G increasing? Where is G decreasing?**

G is increasing where f is positive: $(-1, 4]$. Note that G has a horizontal slope at $x = 2$ but since f is positive on each side of $t = 2$, we say G is increasing at $x = 2$.

G is decreasing where f is negative: $[-4, -1)$.

(c) **Where is G concave up? Where is G concave down?**

G is concave up where f is increasing: $(-2, 0) \cup (2, 3)$.

G is concave down where f is decreasing: $(1, 2) \cup (3, 4]$.

(d) **At what x-value(s) does G have a local maximum? At what x-value(s) does G have a local minimum?**

G has a local maximum where f changes from positive to negative: never.

G has a local minimum where f changes from negative to positive: $x = -1$.

(e) **Find a formula that relates G and H.**

From their definitions, $H(x) = \int_{-3}^{0} f(t)\ dt + G(x) = -2 + G(x)$.

(f) **How would your answers to (b), (c), and (d) change if the questions were about H instead of G?**

They would not change at all because $H'(x) = G'(x)$.

11. **Use sigma notation to express L_{10} and M_{10} as approximations to $\int_{20}^{60} \ln x\ dx$.**

We're subdividing the interval into 10 pieces, so each piece has width $\Delta x = \frac{60 - 20}{10} = 4$.

$$\begin{aligned} L_{10} &= [f(20) + f(24) + f(28) + ... + f(52) + f(56)]\Delta x \\ &= [\ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4 \end{aligned}$$

$$\begin{aligned} M_{10} &= [f(22) + f(26) + f(30) + ... + f(54) + f(58)]\Delta x \\ &= [\ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4 \end{aligned}$$