FINAL EXAM

Math 205
12/10/13 **Name:**

Read all of the following information before starting the exam:

- Show all work clearly in the blue book in order to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- You may work out of order as long as you number each problem clearly.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- Put your test into your blue book before handing in your exam.
- This test has 10 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*5 points*) The matrices A and B shown below are row equivalent.

$$A = \begin{bmatrix} 2 & 4 & 1 & 6 & 1 \\ 3 & 6 & 2 & 7 & 0 \\ 2 & 4 & 0 & 10 & 4 \\ 3 & 6 & 2 & 7 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 1 & 6 & 1 \\ 0 & 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. (*3 pts*) Find a basis for Row A.

b. (*2 pts*) What is the dimension of the subspace W spanned by the columns of A?

2. (*11 points*) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

a. (*8 pts*) Diagonalize A in two ways. (Diagonalization is not unique, so give two possible ways it could be done.)

b. (*3 pts*) Explain how you can determine A^{20} using one of the decompositions from above.

3. (*6 points*) Suppose B is a 5×5 matrix. You know $\lambda = 4$ and $\lambda = -1$ are the only eigenvalues of B. The dimension of the eigenspace of $\lambda = 4$ is 2 and the dimension of the multiplicity of $\lambda = -1$ is 2.

a. (*3 pts*) Is it possible that B is diagonalizable? Why or why not.

b. (*3 pts*) What must be the characteristic equation for B?

4. (*10 points*) Let $\vec{p}_1(x) = 1 + x^2$, $\vec{p}_2(x) = 2 - x + 3x^2$, and $\vec{p}_3(x) = 1 + 2x - 4x^2$.

a. (*6 pts*) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$. Explain.

b. (*4 pts*) Consider the basis $\beta = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ for $\mathbb{P}_2$. Find $\vec{q}$ in $\mathbb{P}_2$ given that $[\vec{q}]_\beta = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.

5. (*9 points*) Orthogonal Diagonalization

a. (*5 pts*) Show that if A is orthogonally diagonalizable, then A^2 is also orthogonally diagonalizable.

b. (*4 pts*) It can be shown that

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ -2 & 2 \end{bmatrix}^{-1}.$$

If possible, give the orthogonal diagonalization of the matrix A.

6. (*6 points*) Let A be an $n \times n$ matrix with eigenvalue λ and $\vec{x}$ be an eigenvector associated with λ. Let S be a nonsingular $n \times n$ matrix and

$$B = SAS^{-1} \text{ and } \vec{y} = S\vec{x}.$$

Show that $\vec{y}$ is an eigenvector of B and determine the eigenvalue of B corresponding to $\vec{y}$.

7. (*13 points*) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$L(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$$

a. (*8 pts*) Find the kernel of L, a basis Ker(L), and the dimension of Ker(L).

b. (*5 pts*) Find the matrix A such that $L(\vec{x}) = A\vec{x}$.

8. (*15 points*) Consider $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 10 \\ 6 \\ -4 \end{bmatrix}$. Let $W = \text{span}\{\vec{v_1}, \vec{v_2}\}$.

a. (*2 pts*) Determine if $\{\vec{v_1}, \vec{v_2}\}$ is an orthogonal set.

b. (*4 pts*) What is the closest vector to $\vec{y}$ in W?

c. (*4 pts*) What is the distance between $\vec{y}$ and the vector you found in part **b.**?

d. (*5 pts*) Find two non-zero vectors in $W^{\perp}$.

9. (*13 points*) Consider the following matrix equation and systems of equations. For each problem, if one solutions exists, then state it. If more than one solution exists, then write the solution in parametric vector form. If no solutions exist, then find the least squares solution instead.

a. (*5 pts*)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}$$

b. (*8 pts*)

$$\begin{aligned} 2x + y + z &= 1 \\ y + z &= 2 \\ -x + 2y + 2z &= -2 \\ 3x - 6z &= 1 \end{aligned}$$

10. (*12 points*) List six statements that are each equivalent to the statement that an $n \times n$ matrix A is invertible. The following concepts should be included, one in each statement: (1) the equation $A\vec{x} = \vec{0}$, (2) rank(A), (3) the columns of A, (4) det A, (5) "one-to-one", and (6) the identity matrix I_n.