

Math 105: Review for Final Exam, Part I - SOLUTIONS

1. Consider the function  $f(x) = \frac{3}{5-2x}$ .

(a) Is this function continuous on the interval  $(-\infty, \infty)$ ? Explain.

No.  $f$  is discontinuous at  $x = 2.5$ , where  $f$  is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of  $f$  on  $[2, 2.01]$ .

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[ \frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122$$

(c) Using the limit definition of the derivative, compute  $f'(x)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\ &= \lim_{h \rightarrow 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6h}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6}{[5 - 2(x+h)](5-2x)} \\ &= \frac{6}{(5-2x)^2} \end{aligned}$$

(d) Find the equation of the tangent line to  $f$  at  $x = 2$ .

We want  $y = mx + b$ .  $m = f'(2) = \frac{6}{(5-2(1))^2} = 6$ , so  $y = 6x + b$ .

[Note that this slope agrees well with our answer from (b) above.]

When  $x = 2$ ,  $y = f(2) = \frac{3}{5-2(2)} = 3$ .

Thus,  $3 = 6 \cdot 2 + b$ , so  $b = -9$  and we have  $y = 6x - 9$ .

2. Given that  $f(0) = 2$ ,  $g(0) = 3$ ,  $f'(0) = 5$ ,  $g'(0) = 7$ , and  $f'(3) = \pi$  compute the following.

(a)  $h'(0)$  if  $h(x) = f(x)g(x)$

$$h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$$

(b)  $j'(0)$  if  $j(x) = \frac{f(x)}{g(x)}$

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

(c)  $k'(0)$  if  $k(x) = f(g(x))$

$$k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$$

3. Compute  $dy/dx$  for each of the following.

(a)  $y = x^5 + 5^x + e^5 + \frac{x}{5} + \frac{5}{x} + \frac{5}{\sqrt[5]{x}} + \ln(5x) + \arctan(5x) + \ln(5) + \sin 5$

$$\begin{aligned} \frac{dy}{dx} &= 5x^4 + (\ln 5)5^x + 0 + \frac{1}{5} - 5x^{-2} + 5 \cdot \frac{-1}{5}x^{-6/5} + \frac{1}{5x} \cdot 5 + \frac{1}{1+(5x)^2} \cdot 5 + 0 + 0 \\ &= 5x^4 + (\ln 5)5^x + \frac{1}{5} - \frac{5}{x^2} - \frac{1}{x^{6/5}} + \frac{1}{x} + \frac{5}{1+25x^2} \end{aligned}$$

(b)  $y = \sqrt[3]{x} \cos(7x^3)$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)$$

(c)  $y = \frac{e^x + e^\pi}{\tan 4 - 7x}$

$$\frac{dy}{dx} = \frac{e^x(\tan 4 - 7x) - (-7)(e^x + e^\pi)}{(\tan 4 - 7x)^2}$$

(d)  $y = \tan(e^{x^2} \arcsin(5x))$

$$\frac{dy}{dx} = \sec^2(e^{x^2} \arcsin(5x)) \cdot e^{x^2} \arcsin(5x) \cdot \left[ x^2 \frac{1}{\sqrt{1-25x^2}} \cdot 5 + 2x \arcsin(5x) \right]$$

(e)  $y^3 + yx^2 + x^2 = 3y^2$

[Implicit Differentiation]

$$\begin{aligned} 3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y \frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y} \end{aligned}$$

4. The rate of change of the temperature  $T(t)$  (in degrees Fahrenheit) of a piece of wedding cake  $t$  hours after it is placed in a very cold freezer is proportional to the the temperature of the cake. The cake's temperature when placed in the freezer is 70 degrees. One hour later, its temperature is 60 degrees.

(a) Write a differential equation whose solution is  $T(t)$ . Your equation may have an unknown constant in it.

Rate of change ( $T'$ ) is (=) proportional to ( $k$ ) temperature of cake ( $T$ ) means  $T' = kT$ .

(b) Solve your differential equation.

The general solution is  $T(t) = Ae^{kt}$ .

What's the value of  $A$ ? When  $t = 0$ , we know  $T = 70$ . That is,  $70 = Ae^0 = A$ , so  $A = 70$ .

What's the value of  $k$ ? When  $t = 1$ , we know  $T = 60$ . That is,  $60 = 70e^{k \cdot 1}$ , or  $6/7 = e^k$ .

To solve for  $k$  we take the ln of each side to get  $\ln(6/7) = k$ .

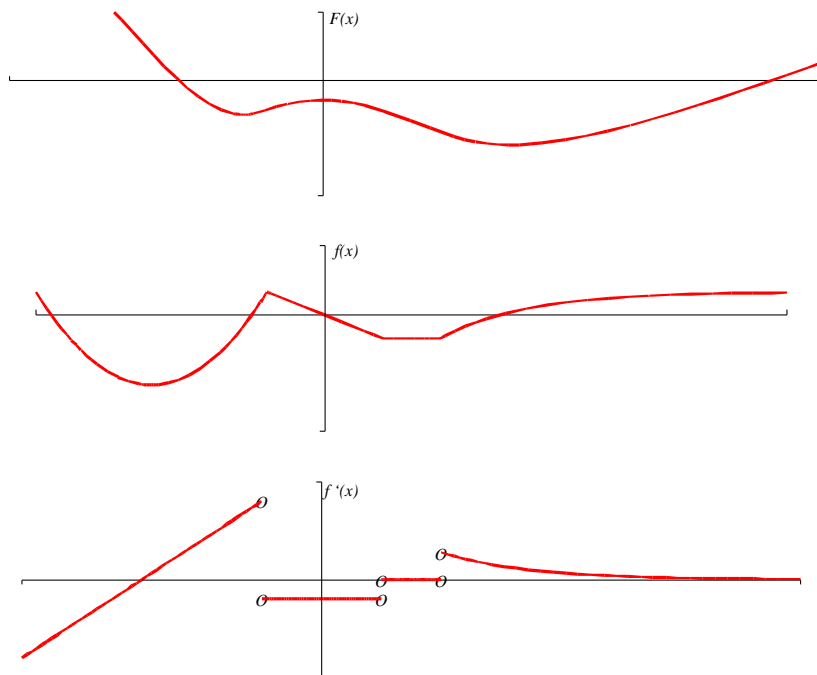
Thus, we have  $T(t) = 70e^{[\ln(6/7)]t}$ , which can be simplified as follows.

$$T(t) = 70e^{[\ln(6/7)]t} = 70[e^{\ln(6/7)}]^t = 70[6/7]^t$$

(c) When will the temperature of the cake reach 0 degrees?

Never. As  $t \rightarrow \infty$ ,  $T$  gets arbitrarily close to 0 but never quite reaches 0. That is, as  $t \rightarrow \infty$ , the graph of  $T(t)$  approaches the  $t$ -axis but never reaches it. If you try to solve the equation  $T(t) = 0$ , you will need to take the logarithm of 0, which is not allowed.

5. Given the graph of  $f$ , sketch a graph of  $f'$  and a graph of  $F$ , an antiderivative of  $f$  such that  $F(0) = -1$ .



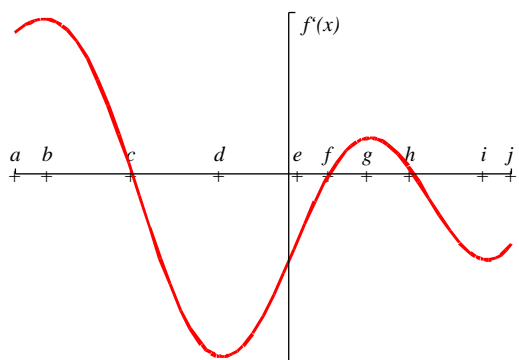
Note: The concave up portion on the left side of the graph of  $f$  is a perfect parabola, so its derivative ( $f'$ ) is linear; since you don't know the equation for  $f$ , your graph of  $f'$  may be concave up/down there.

6. Shown below is a graph of  $f'$  on its entire domain. The graph is NOT  $f$ .

At which  $x$ -value(s)

- (a) does  $f$  have a stationary point?  $c, f, h$
- (b) does  $f$  have a local max?  $c, h$
- (c) does  $f$  have a local min?  $f$
- (d) does  $f'$  have a stationary point?  $b, d, g, i$
- (e) does  $f'$  have a local max?  $b, g$
- (f) does  $f'$  have a local min?  $d, i$
- (g) is  $f$  greatest?  $c$
- (h) is  $f$  least?  $j$
- (i) is  $f'$  greatest?  $b$
- (j) is  $f'$  least?  $d$
- (k) is  $f''$  greatest?  $e$
- (l) is  $f''$  least?  $c$

- (b)  $f$  decreasing?  $(c, f) \cup (h, j]$
- (c)  $f'$  increasing?  $[a, b) \cup (d, g) \cup (i, j]$
- (d)  $f'$  decreasing?  $(b, d) \cup (g, i)$
- (e)  $f$  concave up?  $[a, b) \cup (d, g) \cup (i, j]$
- (f)  $f$  concave down?  $(b, d) \cup (g, i)$



On what interval(s) is

- (a)  $f$  increasing?  $[a, c) \cup (f, h)$

7. Is  $y = 7e^{3x}$  a solution to the differential equation  $y'' + 2y' - 15y = 0$ ? Explain.

A given function  $y$  will be a solution to the differential equation if, when we substitute in  $y''$ ,  $y'$ , and  $y$ , the equation is satisfied (that is, both sides of it are equal).

Since  $y = 7e^{3x}$ , we know that  $y' = 21e^{3x}$  and  $y'' = 63e^{3x}$  from the Chain Rule.

Now we check to see whether our  $y$  satisfies the differential equation.

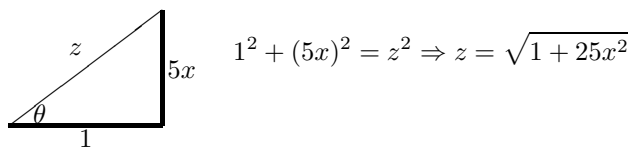
$$\begin{aligned} y'' + 2y' - 15y &\stackrel{?}{=} 0 \\ 63e^{3x} + 2 \cdot 21e^{3x} - 15 \cdot 7e^{3x} &\stackrel{?}{=} 0 \\ 63e^{3x} + 42e^{3x} - 105e^{3x} &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

So, we see that  $y = 7e^{3x}$  is in fact a solution to this differential equation.

8. Rewrite  $\sin(\arctan(5x))$  as an algebraic expression.

Let  $\theta = \arctan(5x)$ . That is,  $\theta$  is the angle whose tangent is  $5x$ .

We draw a triangle for which  $\frac{\text{opposite}}{\text{adjacent}} = \frac{5x}{1} = 5x$ .



$$\sin(\arctan(5x)) = \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{1 + 25x^2}}$$

9. Evaluate the following limits.

Throughout this solution, the symbol  $\star$  will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form  $0/0$ ; this may be  $\stackrel{0/0}{=}$  or  $\stackrel{L'H}{=}$  or  $\stackrel{H}{=}$  or  $\stackrel{0/0}{=}$  or "has the form  $\frac{0}{0}$ , and so, by L'Hopital's Rule, is equal to" or something else. The symbol  $\heartsuit$  will serve the same purpose for the indeterminate form  $\infty/\infty$ .

- (a)  $\lim_{x \rightarrow \infty} \frac{x^2}{\ln x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{1/x} = \lim_{x \rightarrow \infty} 2x^2 = \infty$
- (b)  $\lim_{x \rightarrow 0} \frac{\sin(12x) - 12x}{x^3} \star \lim_{x \rightarrow 0} \frac{12 \cos(12x) - 12}{3x^2} \star \lim_{x \rightarrow 0} \frac{-144 \sin(12x)}{6x} \star \lim_{x \rightarrow 0} \frac{-1728 \cos(12x)}{6} = -288$
- (c)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$
- (d)  $\lim_{x \rightarrow 5} \frac{35 - 7x}{2x - 10} \star \lim_{x \rightarrow 5} \frac{-7}{2} = -3.5$