1. Find the solution to \( \frac{dy}{dx} = \frac{\cos x}{y^2} \) that passes through \((0, 2)\). Separate variables.

\[
\begin{align*}
\frac{y^2}{2} d\frac{y}{y^2} &= \cos x \, dx \\
\frac{1}{3} y^3 &= \sin x + C \\
y^3 &= 3 \sin x + D \\
y &= \sqrt[3]{3 \sin x + D}
\end{align*}
\]

Thus, \( y = \sqrt[3]{3 \sin x + 8} \)

2. Use a second-degree Taylor polynomial to estimate \( \sqrt[3]{28} \). Choose \( x_0 = 27 \) because we know \( \sqrt[3]{27} = 3 \).

\[
\begin{align*}
f(x) &= x^{\frac{1}{3}} \\
f'(x) &= \frac{1}{3} x^{-\frac{2}{3}} \\
f''(x) &= -\frac{2}{9} x^{-\frac{5}{3}} \\
f'''(x) &= \frac{14}{27} x^{-\frac{8}{3}}
\end{align*}
\]

So, \( p_2(x) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2!}(x-27)^2 \)

\[
\begin{align*}
&= 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 \\
&\approx 3 + \frac{1}{27}(28-27) - \frac{1}{2187}(28-27)^2 \\
&\approx \frac{6641}{2187} = 3.0365717...
\end{align*}
\]

3. What is the largest possible error that could have occurred in your previous estimate? \( |E(x) - P_n(x)| \leq \frac{k_{n+1}}{(n+1)!} |x-x_0|^{n+1} \)

\[
|E(x) - P_2(x)| \leq \frac{10}{27 \times 28!} |x-27|^{3} = \frac{5}{521,441}
\]

4. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

(a) \( \int_{1}^{\infty} \frac{7+5 \sin x}{x^2} \, dx \) For \( x \geq 1, 0 \leq \frac{7+5 \sin x}{x^2} \leq \frac{12}{x^2} \)

\[
\begin{align*}
\int_{1}^{\infty} \frac{7+5 \sin x}{x^2} \, dx &\leq \int_{1}^{\infty} \frac{12}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{12}{x} \right]_1^b = 12 \cdot \frac{1}{1} \\
&= \lim_{b \to \infty} \left[ -\frac{12}{b} - \frac{-12}{1} \right] = 12.
\end{align*}
\]

Thus, the original integral converges and \( 12 \) is an upper bound for its value.

(b) \( \int_{1}^{\infty} \frac{1+3x^2+2x^3}{\sqrt{10x^2+x^3}} \, dx \) \( \frac{17}{1} \)

For \( x \geq 1, 0 \leq \frac{2x^3}{3\sqrt{10x^2+17x^2}} \leq \frac{1+3x^2+2x^3}{3\sqrt{10x^2+17x^2}} \) (Made numerator smaller, denominator larger.)

But \( \int_{1}^{\infty} \frac{2x^3}{3\sqrt{10x^2+17x^2}} \, dx = \int_{1}^{\infty} \frac{2x^3}{3\sqrt{27x^2}} \, dx = \frac{2x^3}{3x^4} = \frac{2}{3} \cdot \frac{1}{x} \) and we know \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges. Thus, the original integral must diverge too.
5. Decide if each of the following sequences \( \{a_k\}_{k=1}^{\infty} \) converges or diverges. If a sequence converges, compute its limit.

(a) \( a_k = 3 + \frac{1}{10^k} \), \( 3.1, 3.01, 3.001, 3.0001, \ldots \)

\[ \lim_{k \to \infty} a_k = 3 \Rightarrow \text{sequence converges to 3} \]

(b) \( a_k = (-1)^k \), \( -1, 1, -1, 1, \ldots \)

\[ \lim_{k \to \infty} a_k \text{ does not exist} \Rightarrow \text{sequence diverges} \]

(c) \( a_k = \frac{3 + 5k}{7 + 2k} \)

\[ \lim_{k \to \infty} a_k = \frac{5}{2} \Rightarrow \text{sequence converges to } \frac{5}{2} \]

by L'Hôpital's Rule

6. Circle the appropriate word to complete each of the following statements correctly.

(a) If the individual terms of a series approach 0 (\( \lim_{n \to \infty} a_n = 0 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never). \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, but \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

(b) If the individual terms of a series approach 0.5 (\( \lim_{n \to \infty} a_n = 0.5 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never).

(c) If the individual terms of an alternating series approach 0 (\( \lim_{n \to \infty} a_n = 0 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never). Also need terms to be non-increasing: \( 0 \leq a_1 \leq a_2 \leq a_3 \leq \ldots \).

(d) If the individual terms of a geometric series approach 0 (\( \lim_{n \to \infty} a_n = 0 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never).

\[ a + ar + ar^2 + \ldots \text{ converges if } |r| < 1, \text{ which means } \lim_{n \to \infty} a r^n = 0. \]

(e) If the ratio of the terms of a series approaches 1 (\( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never).

When this limit = 1, Ratio Test is inconclusive.

(f) If the ratio of the terms of a series approaches 0.5 (\( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0.5 \)), then the series \( \sum_{n=1}^{\infty} a_n \) will converge (always/sometimes/never).

This is the Ratio Test.

(g) If a series has all positive terms, then it will converge to 0 (always/sometimes/never). We can't add up positive terms and get a sum of zero.

7. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) \( 3.1 + 3.01 + 3.001 + 3.0001 + \ldots \) diverges. By nth term test: \( \lim_{k \to \infty} a_k = 3 \neq 0 \).

(b) \( 1 + 1/2 + 1/3 + 1/4 + \ldots \) diverges. This is the harmonic series. The Integral Test shows it diverges because \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges.

(c) \( 5 - 5/3 + 5/9 - 5/27 + \ldots \) This is a geometric series with \( r = -\frac{1}{3} \), so it converges to \( \frac{a}{1-r} = \frac{5}{1-\left(-\frac{1}{3}\right)} = \frac{15}{4} \).
8. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

a) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \)

**Alternating Series Test:**
1) terms alternate in sign
2) \( \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{3}} \geq \frac{1}{\sqrt{4}} \geq \ldots \geq 0 \)
3) \( \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0 \) \( \Rightarrow \) series **converges**

We also know the series is bounded by any 2 consecutive partial sums, so

\( S_1 = \frac{-1}{\sqrt{2}} \) is a lower bound while \( S_2 = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \) is an upper bound.

**Ratio Test:**
b) \( \sum_{n=1}^{\infty} \frac{(2n)!}{3^{n}(n!)^2} \)

\[
\lim_{n \to \infty} \frac{(2(n+1))!}{3^{n+1}((n+1)!)^2} \cdot \frac{3^n (n!)^2}{(2n)!} = \lim_{n \to \infty} \frac{(2n+2)(2n+1)}{3(n+1)^2} = \lim_{n \to \infty} \frac{4n^2 + 6n + 2}{3n^2 + 6n + 3} = \frac{4}{3} \text{ by l'Hôpital's Rule.}
\]
Since \( \frac{4}{3} > 1 \) series **diverges**

**nth term test:**
\( \lim_{n \to \infty} \left( \frac{1}{100} + \frac{1}{n^3} \right) = \frac{1}{100} \neq 0 \) \( \Rightarrow \) series **diverges**

d) \( \sum_{n=1}^{\infty} \frac{\sqrt{9n^3 + 5n^6}}{12n^5 + 3} \)

\[
\sum_{n=1}^{\infty} \frac{\sqrt{9n^3 + 5n^6}}{12n^5 + 3} > \sum_{n=1}^{\infty} \frac{3n^4}{15n^5} = \frac{3n^4}{15n^5} = \frac{1}{5} \cdot \frac{1}{n}
\]

Since we know that \( \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, our series must **diverge** too.

e) \( \sum_{n=2}^{\infty} \frac{1}{n(ln(n))^2} \)

\[
\int_{2}^{\infty} \frac{1}{x(\ln{x})^2} \, dx = \lim_{b \to \infty} \left[ \frac{-1}{\ln{x}} \right]_{2}^{b} = \lim_{b \to \infty} \left( \frac{-1}{\ln{b}} - \frac{-1}{\ln{2}} \right) = \frac{1}{\ln{2}}
\]

Since the integral converges, the series **converges** too.

Further, we know that \( \int_{2}^{\infty} f(x) \, dx \leq \sum_{n=2}^{\infty} f(n) \)
so \( \frac{1}{\ln{2}} \) is a lower bound and \( \frac{1}{2(\ln(2))^2} + \frac{1}{\ln{2}} \) is an upper bound.

9. Does the first series from the previous problem converge absolutely or conditionally?
\( \sum_{n=1}^{\infty} \frac{1}{3\sqrt{n+1}} \) diverges by the Comparison Test or Integral Test.
10. Compute the radius and interval (including endpoints) of convergence for \( \sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 5^n} \).

\[
\lim_{n \to \infty} \left| \frac{(x+3)^n}{n \cdot 5^n} \right| = \lim_{n \to \infty} \left| \frac{(x+3)^n}{x+3} \cdot \frac{1}{5^n} \right| = \left| \frac{x+3}{5} \right|
\]

So, need \( | \frac{x+3}{5} | < 1 \), i.e., \(-1 < \frac{x+3}{5} \leq 1 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2 \).

Check endpoints:

- \( x = 2 \):
  \[
  \sum_{n=1}^{\infty} \frac{(2+3)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges (harmonic).}
  \]

- \( x = -8 \):
  \[
  \sum_{n=1}^{\infty} \frac{(-8+3)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ which converges by A.S.T.}
  \]

Thus, interval is \([-8 \leq x \leq 2]\). Radius = 5.

11. Find the complete Taylor series (in summation notation) for \( f(x) = \ln(1-x) \) about \( x = 0 \) and determine its interval of convergence.

\[
f(x) = \ln(1-x), \quad f(0) = 0, \quad f'(0) = -1, \quad f''(0) = -\frac{1}{1-x}, \quad f'''(0) = -\frac{2}{(1-x)^2}, \quad f^{(4)}(0) = -\frac{6}{(1-x)^3}
\]

Series:

\[
0 - x - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{6x^4}{4} - \ldots = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots
\]

So, need \( |x| < 1 \) or \(-1 < x < 1 \).

Endpoints:

- \( x = 1 \) \( \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic)} \)
- \( x = -1 \) \( \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ which converges by A.S.T.} \)

So, interval is \([-1 \leq x < 1]\).

12. Write the complete series equal to \( \int_0^1 e^{-x^2} \, dx \) and show that it converges.

We know \( e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots \) so \( e^{-X^2} = 1 - X^2 + \frac{X^4}{2!} - \frac{X^6}{3!} + \ldots \)

Thus, \( \int_0^1 e^{-x^2} \, dx = \int_0^1 (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots) \, dx = \left[ (1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \ldots) \right] (0 - 0 + 0 - 0 - \ldots) \)

This converges by the Alternating Series Test. 1) terms alternate 2) \( \left| \frac{1}{3} \cdot \frac{1}{5 \cdot 2!} \right| < \frac{1}{7 \cdot 3!} \) ... 3) \( \lim_{n \to \infty} a_n = 0 \).