Final Exam
MATH 205, Fall 2014

Name: ____

Instructions: Please answer as many of the following questions as possible. Show all of your work and give complete explanations when requested. Write your final answer clearly. No calculators or cell phones are allowed.

This exam has 8 problems and 150 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	24	
2	20	
3	20	
4	22	
5	8	
6	14	
7	20	
8	22	
TOTAL	**150**	

1. (24 points) Consider the following *augmented* matrix for the matrix equation $A\mathbf{x} = \mathbf{b}$ in reduced echelon form:

$$\texttt{rref}\Big(\big[\, A \quad \mathbf{b} \,\big]\Big) = \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) (10 points) Write the solution to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

   SOLUTION: From the reduced echelon form the solution to the matrix equation is

$$\begin{aligned} x_1 + 3x_3 &= 5 \\ x_2 - 2x_3 &= 3 \\ x_4 &= 1. \end{aligned}$$

   Then x_3 is a free variable and in parametric vector form the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad s \text{ in } \mathbb{R}.$$

   (b) (8 points) Find all solutions to the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

   SOLUTION: From part (a), the solution to $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad s \text{ in } \mathbb{R}.$$

   (c) (6 points) Does the matrix equation $A\mathbf{x} = \mathbf{c}$ have a solution for *every* $\mathbf{c}$ in $\mathbb{R}^3$? *Give a one sentence explanation supporting your answer.*

   SOLUTION: Yes, the matrix equation $A\mathbf{x} = \mathbf{c}$ has a solution for every $\mathbf{c}$ in $\mathbb{R}^3$ because in reduced echelon form the matrix A has a pivot in every row.

2. (20 points)

(a) (12 points) Decide whether each of the following matrices is diagonalizable. Justify your answer in each case. You do not need to find P, D or P^{-1}.

$$A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

Solution:

- The matrix A is diagonalizable. The matrix is in triangular form, so its eignevalues are the diagonal entries. Thus, A has eignevalues $1, 3$ and 0. Since A is 3×3 and has 3 distinct eigenvalues, it is diagonalizable.
- The matrix B is diagonalizable. To see this, first we find the eigenvalues by solving the characteristic equation $\det\big(B - \lambda I_3\big) = 0$. Using a cofactor expansion down the first column, the characteristic equation is

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0.$$

Then the eigenvalues for B are $\lambda = 1$ (with multiplicity 2), and $\lambda = 2$ (with multiplicity 1). It will be sufficient to determine if B is diagonalizable by examining the eigenspace associated with $\lambda = 1$.

For $\lambda = 1$ we find the nullspace of the matrix $B - I_3 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

The reduced echelon form of $B - I_3$ is $\begin{bmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so a basis for Nul $\big(B - I_3\big)$ is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Now the eigenspace associated with eigenvalue $\lambda = 1$ has dimension 2, equal to the multiplicity of the eigenvalue. The remaining eignevalue of B has multiplicity 1 so we can conclude that B is diagonalizable.

(b) (8 points) Find E^{2014} where $E = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: To solve this problem I will diagonalize E. The matrix E is in triangular form so it has eignvalues $\lambda = 1$ and $\lambda = -1$. A basis for the eigenspace corresponding to $\lambda = 1$ is given by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = -1$ is given by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then $E = PDP^{-1}$, with

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Finally, $E^{2014} = PD^{2014}P^{-1}$. Since D is a diagonal matrix,

$$D^{2014} = \begin{bmatrix} 1^{2014} & 0 \\ 0 & -1^{2014} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Then $E^{2014} = PI_2P^{-1} = PP^{-1} = I_2$, or in other words

$$E^{2014} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. (20 points)

(a) (10 points) Consider the 3×3 matrix

$$A = \begin{bmatrix} 0 & 1 & k \\ 2 & k & -6 \\ 2 & 7 & 4 \end{bmatrix}.$$

For what values of the constant k is the matrix A invertible?

SOLUTION: I will use the criteria that A is invertible if and only if $\det A \neq 0$. Using a cofactor expansion along the first row,

$$\begin{aligned} \det A &= -\big((2)(4) - (-6)(2)\big) + k\big((2)(7) - (k)(2)\big) \\ &= -20 + 14k - 2k^2 \\ &= -2\big(k^2 - 7k + 10\big) \\ &= -2(k-5)(k-2). \end{aligned}$$

Then $\det A = 0$ if and only if $k = 2, 5$. Thus, A is invertible for all real numbers $k \neq 2, 5$.

(b) (10 points) Consider the 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers and $a \neq 0$. Find all values d such that $\operatorname{rank} B = 1$.

SOLUTION: To begin, row reduce the matrix to reduced echelon form:

$$\texttt{rref}(B) = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix}.$$

The matrix above is defined since $a \neq 0$. Then B has rank 1 as long as there is no pivot in the second row (or column), that is when $d - \frac{bc}{a} = 0$ or equivalently, when $d = \frac{bc}{a}$.

4. (22 points) Let $W = \text{Span}\{\mathbf{x_1}, \mathbf{x_2}\}$, where $\mathbf{x_1} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

(a) (10 points) Use the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ for W.

SOLUTION: First, let $\mathbf{u_1} = \mathbf{x_1}$ and define $W_1 = \text{Span}\{\mathbf{u_1}\}$.
Next, let $\mathbf{u_2} = \mathbf{x_2} - \text{proj}_{W_1}\mathbf{x_2}$. Then we calculate

$$\begin{aligned} \mathbf{u_2} &= \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} \\ &= \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} - \frac{-5}{5}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Then $\{\mathbf{u_1}, \mathbf{u_2}\} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(b) (12 points) Let $\mathbf{y} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$. Decompose $\mathbf{y}$ as $\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$, where $\widehat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

SOLUTION: Write $\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$ where $\widehat{\mathbf{y}} = \text{proj}_W\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}}$. Using the orthogonal basis for W found in part (a),

$$\begin{aligned} \widehat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} + \frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\mathbf{u_2} \\ &= \frac{6}{5}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + \frac{-9}{6}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{5} \\ -\frac{39}{10} \\ \frac{3}{2} \end{bmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{z} &= \mathbf{y} - \widehat{\mathbf{y}} \\
&= \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{9}{5} \\ -\frac{39}{10} \\ \frac{3}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{1}{2} \end{bmatrix}.
\end{aligned}$$

5. (8 points) Let $A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ x & y \end{bmatrix}$. Find x and y knowing that A is an orthonormal matrix.

SOLUTION: In order for A to be an orthonormal matrix, its columns must be orthonormal vectors. That is,

$$\begin{bmatrix} \frac{1}{\sqrt{5}} \\ x \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{5}} \\ y \end{bmatrix} = 0$$

and

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{5}} \\ x \end{bmatrix} \right\| = 1, \quad \text{and} \quad \left\| \begin{bmatrix} \frac{2}{\sqrt{5}} \\ x \end{bmatrix} \right\| = 1.$$

The first condition says that $\frac{2}{5} + xy = 0$, or $xy = -\frac{2}{5}$. The second condition tells me that $\sqrt{\frac{1}{5} + x^2} = 1$, or that $x^2 = \frac{4}{5}$ and similarly $y^2 = \frac{1}{5}$. Then there are two possibilities, either

$$x = \frac{2}{\sqrt{5}} \quad \text{and} \quad y = -\frac{1}{\sqrt{5}}$$

or

$$x = -\frac{2}{\sqrt{5}} \quad \text{and} \quad y = \frac{1}{\sqrt{5}}.$$

6. (14 points) Consider the 2×2 matrix $B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$.

   (a) (8 points) Find a basis for $(\text{Row } B)^{\perp}$.

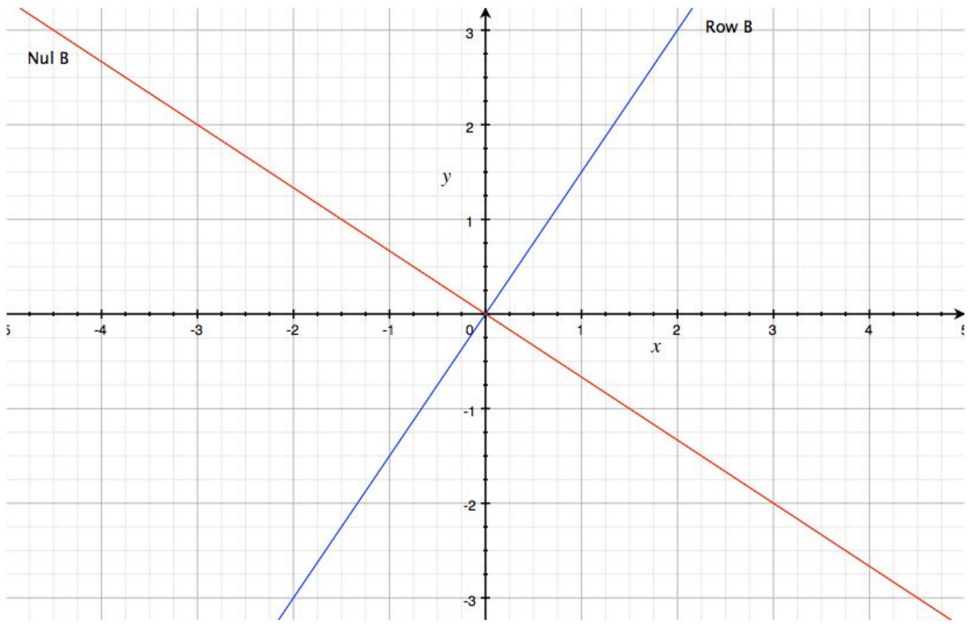
   SOLUTION: We know that $(\text{Row } B)^{\perp} = \text{Nul } B$, so first row reduce the matrix B to reduced echelon form:

$$\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

   Then a basis for $\text{Nul } B$ is $\left\{ \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} \right\}$.

   (b) (6 points) On the coordinate axis below, plot $\text{Row } B$ and $\text{Nul } B$.

   SOLUTION:



a

7. (20 points)

(a) (10 points) Is the set of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with $\|\mathbf{v}\| = 1$ a subspace of $\mathbb{R}^n$? *Write your answer in complete sentences.*

SOLUTION: No, the set of all unit vectors in $\mathbb{R}^n$ is not a subspace of $\mathbb{R}^n$. In fact, this set violates all three conditions of a subspace. First of all, the zero vector is not a unit vector since $\|\mathbf{0}\| = 0$. Secondly, the sum of two unit vectors is not a unit vector. And finally, since for any c in $\mathbb{R}$, $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$, scalar multiples of unit vectors are not unit vectors.

(b) (10 points) Find a vector $\mathbf{q}(t)$ such that the set

$$\left\{2t+1, t^2-1, \mathbf{q}(t)\right\}$$

is a basis for $\mathbb{P}_2$. Justify your answer.

SOLUTION: Let $\mathbf{q}(t) = at^2 + bt + c$ be an arbitrary polynomial in $\mathbb{P}_2$. Using the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$, we can map each polynomial in the set to its coordinate vector, yielding the three vectors

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

To ensure that the three polynomials form a basis for $\mathbb{P}_2$, we put the three coordinate vectors into a matrix and choose a, b, c so that there is a pivot in every column and every row. We have

$$M = \begin{bmatrix} 1 & -1 & c \\ 2 & 0 & b \\ 0 & 1 & a \end{bmatrix}, \qquad \texttt{echelon form}(M) = \begin{bmatrix} 1 & -1 & c \\ 0 & 1 & a \\ 0 & 0 & -2a+b-2c \end{bmatrix}.$$

There will be a pivot in every row and column provided $-2a + b - 2c \neq 0$. Then any polynomial $\mathbf{q}(t) = at^2 + bt + c$ satisfying $-2a + b - 2c \neq 0$ will satisfy the conditions of the problem. Some examples of correct answers include:

$$\mathbf{q}(t) = t^2;$$
$$\mathbf{q}(t) = t;$$
$$\mathbf{q}(t) = 1.$$

8. (22 points) Let $S : \mathbb{P}_2 \to \mathbb{P}_2$ be defined as $S\big(\mathbf{p}(t)\big) = t\mathbf{p}'(t) + \mathbf{p}'(0)$.

(a) (10 points) Show that S is a linear transformation.

SOLUTION: First we show that $S\big((\mathbf{p}+\mathbf{q})(t)\big) = S\big(\mathbf{p}(t)\big) + S\big(\mathbf{q}(t)\big)$ for all $\mathbf{p}(t), \mathbf{q}(t)$ in $\mathbb{P}_2$:

$$\begin{aligned} S\big((\mathbf{p}+\mathbf{q})(t)\big) &= t\big(\mathbf{p}+\mathbf{q}\big)'(t) + \big(\mathbf{p}+\mathbf{q}\big)'(0) \\ &= t\big(\mathbf{p}'(t) + \mathbf{q}'(t)\big) + \mathbf{p}'(0) + \mathbf{q}'(0) \\ &= t\mathbf{p}'(t) + \mathbf{p}'(0) + t\mathbf{q}'(t) + \mathbf{q}'(0) \\ &= S\big(\mathbf{p}(t)\big) + S\big(\mathbf{q}(t)\big). \end{aligned}$$

Next, let k in $\mathbb{R}$ and we must show that $S\big((k\mathbf{p})(t)\big) = kS\big(\mathbf{p}(t)\big)$:

$$\begin{aligned} S\big((k\mathbf{p})(t)\big) &= t\big(k\mathbf{p}\big)'(t) + \big(k\mathbf{p}\big)'(0) \\ &= tk\mathbf{p}'(t) + k\mathbf{p}'(0) \\ &= k\big(t\mathbf{p}'(t) + \mathbf{p}'(0)\big) \\ &= kS\big(\mathbf{p}(t)\big). \end{aligned}$$

(b) (8 points) Find a basis for $\ker S$.

SOLUTION: Take an arbitrary element of $\mathbb{P}_2$, say $at^2 + bt + c$. Then we aim to find a, b, c so that $S\big(at^2 + bt + c\big) = 0$. First,

$$\begin{aligned} S\big(at^2 + bt + c\big) &= t(2at + b) + b \\ &= 2at^2 + bt + b. \end{aligned}$$

Then setting the polynomial $2at^2 + bt + b = 0$ (the zero polynomial), we see immediately that $a = b = 0$. The constant term c is free, therefore for any constant c, $S\big(c\big) = 0$. Then a basis for the kernel of S is the polynomial $\{1\}$.

(c) (4 points) Is S one-to-one? *Give a one sentence explanation of your answer.*

SOLUTION: The transformation S is not one-to-one because the kernel has non-trivial elements.