Final Exam
MATH 205, Fall 2014

Name: ____

Instructions: Please answer as many of the following questions as possible. Show all of your work and give complete explanations when requested. Write your final answer clearly. No calculators or cell phones are allowed.

This exam has 8 problems and 150 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	24	
2	20	
3	20	
4	22	
5	8	
6	14	
7	20	
8	22	
TOTAL	**150**	

1. (24 points) Consider the following *augmented* matrix for the matrix equation $A\mathbf{x} = \mathbf{b}$ in reduced echelon form:

$$\texttt{rref}\big(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}\big) = \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) (10 points) Write the solution to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

(b) (8 points) Find all solutions to the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

(c) (6 points) Does the matrix equation $A\mathbf{x} = \mathbf{c}$ have a solution for *every* $\mathbf{c}$ in $\mathbb{R}^3$? *Give a one sentence explanation supporting your answer.*

2. (20 points)

(a) (12 points) Decide whether each of the following matrices is diagonalizable. Justify your answer in each case. You do not need to find P, D or P^{-1}.

$$A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

(b) (8 points) Find E^{2014} where $E = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

3. (20 points)

(a) (10 points) Consider the 3×3 matrix

$$A = \begin{bmatrix} 0 & 1 & k \\ 2 & k & -6 \\ 2 & 7 & 4 \end{bmatrix}.$$

For what values of the constant k is the matrix A invertible?

(b) (10 points) Consider the 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers and $a \neq 0$. Find all values d such that $\operatorname{rank} B = 1$.

4. (22 points) Let $W = \text{Span}\{\mathbf{x_1}, \mathbf{x_2}\}$, where $\mathbf{x_1} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

(a) (10 points) Use the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ for W.

(b) (12 points) Let $\mathbf{y} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$. Decompose $\mathbf{y}$ as $\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$, where $\widehat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

5. (8 points) Let $A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ x & y \end{bmatrix}$. Find x and y knowing that A is an orthonormal matrix.

6. (14 points) Consider the 2×2 matrix $B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$.

   (a) (8 points) Find a basis for $\left(\text{Row}\, B\right)^{\perp}$.

   (b) (6 points) On the coordinate axis below, plot $\text{Row}\, B$ and $\text{Nul}\, B$.

7. (20 points)

   (a) (10 points) Is the set of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with $\|\mathbf{v}\| = 1$ a subspace of $\mathbb{R}^n$? *Write your answer in complete sentences.*

   (b) (10 points) Find a vector $\mathbf{q}(t)$ such that the set

$$\left\{2t+1, t^2-1, \mathbf{q}(t)\right\}$$

   is a basis for $\mathbb{P}_2$. Justify your answer.

8. (22 points) Let $S : \mathbb{P}_2 \to \mathbb{P}_2$ be defined as $S\big(\mathbf{p}(t)\big) = t\mathbf{p}'(t) + \mathbf{p}'(0)$.

(a) (10 points) Show that S is a linear transformation.

(b) (8 points) Find a basis for $\ker S$.

(c) (4 points) Is S one-to-one? *Give a one sentence explanation of your answer.*