Math 205A Quiz 8, page 1 November 30, 2007 NAME *suggested sol'ns*

1. Let $A \in M_{n\times n}$. We say A is a *diagonalizable* matrix if and only if there exist two matrices P and D both in $M_{n\times n}$ such that:

1A. P is what kind of a matrix? *invertible* **1B.** D is what kind of a matrix? *diagonal*

1C. A equals what product in terms of P and D? PDP^{-1}

2. If $A = \begin{bmatrix} 65 & 6 & -30 \\ -20 & 3 & 10 \\ 120 & 12 & -55 \end{bmatrix}$, then A has eigenvalues 3 and 5. An eigenvector for 3 is $\begin{bmatrix} -3 \\ 1 \\ -6 \end{bmatrix}$, and 5 has multiplicity 2. Use this information and the Diagonalization Theorem to diagonalize the matrix A. (Just find P and D.)

we need to find a basis for the eigenspace of $\lambda = 5$. It must have dimension 2 or A will not be diagonalizable. Now, the eigenspace of $\lambda = 5$ is the null space of $A - 5I$. We find

$$A - 5I = \begin{bmatrix} 60 & 6 & -30 \\ -20 & -2 & 10 \\ 120 & 12 & -60 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 10 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 10x_1 = -x_2 + 5x_3 \text{ where } x_2 \,\&\, x_3 \text{ are free, ie}$$

$$(A-5I)\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2/10 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1/10 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for eigenval $\lambda = 5$ is $\left\{\begin{bmatrix} -1/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}\right\}$ or, if you don't like fractions, $\left\{\begin{bmatrix} -1 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\}$ Any basis will do.

$P = \begin{bmatrix} -3 & -1 & 1 \\ 1 & 10 & 0 \\ -6 & 0 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (there are other sol'ns, of course)

3. Give an example of a matrix $S \in M_{2\times 2}$ which *is* invertible, but is *not* diagonalizable, and explain why S has these two properties.

The easiest way to do this is to find one which has non zero determinant "dee"?
AND which has a characteristic poly. with no real roots (so no eigenvalues, so no "D")
An example is $\begin{bmatrix} 1 & -10 \\ 1 & 1 \end{bmatrix}$; det is 11 so S^{-1} exists, but its char. poly is $(1-\lambda)(1-\lambda)+10$
$= \lambda^2 - 2\lambda + 1 + 10$
$= \lambda^2 - 2\lambda + 11$, which has NO real roots.

4. Suppose $Q \in M_{2\times 2}$ and Q has the form $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$ where a is some real number.

4A. Find the determinant of Q. det= 0

$$|Q| = a(-a) - (-a)(a) = -(a^2) + a^2 = 0$$

4B. Is Q invertible? Circle one: Y **N**

4C. Find and simplify the characteristic polynomial of Q. polynomial is λ^2 (NOT "$\lambda^2 = 0$")

$$\begin{vmatrix} a-\lambda & a \\ -a & -a-\lambda \end{vmatrix} = (a-\lambda)(-a-\lambda) + a^2 = -\cancel{a^2} + \lambda a - \cancel{\lambda a} + \lambda^2 + \cancel{a^2} = \lambda^2$$

4D. Find the eigenvalues of Q along with their multiplicities. eigvals & multiplicities $\lambda = 0$ with mult. 2

4E. Find a basis for the eigenspace of each eigenvalue. most everyone proceeded like this: with $\lambda = 0$,

$$\begin{bmatrix} a-\lambda & a \\ -a & -a-\lambda \end{bmatrix} = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix} \sim \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \overset{(*)}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 \text{ free} \end{matrix} \Rightarrow \vec{x} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ so the eigen-

4F. Is Q diagonalizable? Why or why not? space of $\lambda = 0$ has basis $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$. Since dim of this is ONE but $\lambda = 0$ has multiplicity 2, Q is NOT diagonalizable... FINE EXCEPT!!!

EXCEPT: if $a = 0$!!! Then $(*)$ is WRONG! and $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is diagonalizable (since it's diagonal) and the eigenspace is ALL of $\mathbb{R}^2$! $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$