1. Let $A = \begin{bmatrix} 6 & 0 & 1 \\ -2 & 4 & -1 \\ -2 & 0 & 3 \end{bmatrix}$; then A is diagonalizable and has only two eigenvalues, 4 and 5. Find matrices P and D which represent a diagonalization of A. Even though only one of them is used in the diagonalization, find *both* P^{-1} and D^{-1}.

You may find it useful that $\begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & -1 \\ -2 & 0 & -1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ -2 & 0 & -2 \end{bmatrix}$

Show all your work.

Let's find the nullspace of $A-\lambda I$ where...

$$\lambda = 4: \quad A - 4I = \begin{bmatrix} 6-4 & 0 & 1 \\ -2 & 4-4 & -1 \\ -2 & 0 & 3-4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & -1 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the soln's of $A-4I=\vec{0}$ are $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ where x_2 & x_3 are free,

a basis for this nullspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 5$ similarly, a basis for the null space of $A-5I$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

there are variations on the following answers, since D isn't unique and neither are the bases above, nor is the order in which we write the 2 vectors for $\lambda=4$ when making P. so...

One answer:

$P = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 0 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

2. Find a 2×2 matrix A that has $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ as eigenvectors with associated eigenvalues 3 and 1, respectively. Show all your work, including any matrices you create in order to do this problem. CIRCLE your final matrix A.

Let's "make" $P = \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$; then $P^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -4 \\ -2 & 2 \end{bmatrix}$

and $A = PDP^{-1} = \begin{bmatrix} 11 & -8 \\ 10 & -7 \end{bmatrix}$

ANOTHER WAY to do this problem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then

$$A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} 2a + 2b = 6 \\ 2c + 2d = 6 \end{cases}$$

AND $$A\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} 4a + 5b = 4 \\ 4c + 5d = 5 \end{cases}$$

this gives 4 eqn's in 4 unknowns, resulting in the augmented matrix:

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 6 \\ 0 & 0 & 2 & 2 & 6 \\ 4 & 5 & 0 & 0 & 4 \\ 0 & 0 & 4 & 5 & 5 \end{array}\right]$$

which has RREF $\left[\begin{array}{c|c} I_4 & \begin{matrix} 11 \\ -8 \\ 10 \\ -7 \end{matrix} \end{array}\right]$, ie $\begin{cases} a = 11 \\ b = -8 \\ c = 10 \\ d = -7 \end{cases} \Rightarrow A = \begin{bmatrix} 11 & -8 \\ 10 & -7 \end{bmatrix}$

3. Mark each statement TRUE if it is always true, and FALSE if there are counter examples. Let A be an $n \times n$ square matrix.

a. A^{-1} exists $\Rightarrow$ A is diagonalizable. $\boxed{F}$

A^{-1} exists does NOT mean the roots of the char. poly will all be real, or that "multiplicities match" (eigenspace dimensions = eig. val. mults)

b. A^{-1} exists $\Rightarrow$ A does not have 0 as an eigenvalue. $\boxed{T}$

c. A is diagonalizable $\Rightarrow$ A^{-1} exists. $\boxed{F}$

d. If 0 is not an eigenvalue of A then A is not singular. $\boxed{T}$

A^{-1} exists means the only solution to $A\vec{x} = 0\vec{x}$ is $\vec{x} = \vec{0}$

to say 0 is NOT an eigenvalue means $A\vec{x} = 0\vec{x}$ has only the trivial soln, so its cols. are L.I., so it's invertible, i.e. non singular

replace the "3" with a "0" in problem 2. A will be diagonalizable, but A^{-1} won't exist.