TEST 2B

Math 205
11/18/13

Name: Key
by writing my name I swear by the honor code

Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can and should be reduced into RREF with calculator without showing steps unless otherwise indicated). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and one Bonus and is worth 95 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*20 points*) Consider $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ -1 & 2 & 4 \end{bmatrix}$

a. (*4 pts*) Find the characteristic equation $c_A(x)$ for A. Find the eigenvalues of A.

$C_A(x) \Rightarrow (3-\lambda)^2(-\lambda) = 0$

eigenvalues are $\lambda = 3, 0$

b. (*4 pts*) Find the characteristic equation $c_B(x)$ for B. Find the eigenvalues of B.

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 2 & -1-\lambda & -2 \\ -1 & 2 & 4-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)[(-1-\lambda)(4-\lambda)+4]$$

$-4+\lambda-4\lambda+\lambda^2+4$

$(3-\lambda)(\lambda^2-3\lambda)$

$c_B(x) = (3-\lambda)(\lambda)(\lambda-3) = 0$

eigenvalues are $\lambda = 3, 0$

c. (*5 pts*) Determine whether A is diagonalizable. If so, find P and D such that $A = PDP^{-1}$.

$\lambda = 3$ $\begin{pmatrix} 0 & 5 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ The Nullspace is 1-dimensional

A is not diagonalizable

d. (*5 pts*) Determine whether B is diagonalizable. If so, find P and D such that $B = PDP^{-1}$.

$\lambda = 3$ $\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\sim$ Basis $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 0$ $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ -1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\sim$ Basis $= \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ $P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

e. (*2 pts*) What is $c_A(A)$ and what is $c_B(B)$? (You should NOT have to calculate this to answer this question).

$C_A(A) = O_{3\times 3}$ $\quad$ $C_B(B) = O_{3\times 3}$

2. (*15 points*) Given $A = \begin{bmatrix} 0 & 1 & -2 & 2 & 0 \\ -1 & 3 & 0 & 1 & 6 \\ 8 & -1 & 3 & 5 & 1 \end{bmatrix} \to_{rref} \begin{bmatrix} 1 & 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$. Find a basis for the following.

a. (*5 pts*) Col(A)

$$= \left\{ \begin{bmatrix} 0 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}$$

b. (*5 pts*) Nul(A)

$$= \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c. (*5 pts*) Row(A)

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

3. (*9 points*) Given an $m \times n$ matrix A. Prove (verify in generality) that Nul(A) is a subspace of $\mathbb{R}^n$.

1) $\vec{0} \in Nul(A)$ because $A\vec{0} = \vec{0}$

2) $\vec{x}, \vec{y}$ are in $Nul(A)$ then

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$ and $\vec{x}+\vec{y} \in Nul(A)$

3) $\vec{x}$ is in $Nul(A)$, c a scalar then

$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}$ and $c\vec{x} \in Nul(A)$.

Nullspace is a subspace of $\mathbb{R}^n$.

4. (*10 points*)

- __T__ If A is a 2×2 matrix and has 2 distinct eigenvalues, then A must be diagonalizable.
- __T__ If matrices A and B have the same reduced row echelon form then Row(A) = Row(B).
- __T__ If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that H = Col(A).
- __T__ A subspace is also a vectorspace.
- __F__ A linearly independent set in a subspace H is a basis for H.

5. (*8 points*) Consider vectors of the form $\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}$.

a. (*3 pts*) Show that the vectors form a subspace of $\mathbb{R}^4$.

The vectors are a span and a span is a subspace.

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$$

b. (*5 pts*) Find a basis for the set.

$$\begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

6. (*15 points*) Short answer.

a. (*5 pts*) Prove that if 0 is an eigenvalue of A, then A is not invertible (or an equivalent statement opposition to the IMT).

If $\lambda = 0$ is a eigenvalue of A, then

$(A - \vec{0}I)\vec{x} = A\vec{x}$ has non-trivial solutions.

The IMT states that A must be non-singular.

b. (*5 pts*) Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix such that $AB = 0$. Show that rank (A) + rank$(B) \leq n$ [Hint: one of the four subspaces Nul(A), Col(A), Nul(B), Col(B) is contained in one of the other three spaces.)

Recall rank(A) + nullity$(A) = n$

If $AB = 0$ then the columns of B are in Nul(A)

so $\text{Col}(B) \subseteq \text{Nul}(A)$

$\text{rank}(B) = \dim(\text{Col}(B)) \leq \text{Nullity}(A)$

So $\text{rank}(A) + \text{rank}(B) \leq n$.

c. (*5 pts*) Use coordinate vectors to test the whether the set of polynomials span $\mathbb{P}_2$.

$$\{1 - 3t + 5t^2, -3 + 5t - 7t^2, -4 + 5t - 6t^2, 1 - t^2\}$$

Use $\beta = \{1, t, t^2, t^3\}$

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/4 & -5/4 \\ 0 & 1 & 7/4 & -3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ No!

Not a pivot in every row.

7. (*18 points*) Let $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be a basis for $M_{2\times 2}(\mathbb{R})$.

a. (*7 pts*) Use coordinate vectors to prove whether $\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yes, it is a basis.

Consider the transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ where $T(A) = -A^T$. Recall an element in $M_{2\times 2}(\mathbb{R})$ can be represented as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

b. (*3 pts*) What is the Ker(T)?

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} \text{ if } = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then $\text{Ker}(T) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The transformation T has $\lambda = 1$ and $\lambda = -1$ as eigenvalues. (ie. There exists vectors $\vec{x}$ such that $T(x) = \vec{x}$ or $T(\vec{x}) = -\vec{x}$.)

c. (*4 pts*) Without finding the matrix associated with T find the matrices in the eigenspace associated with $\lambda = 1$, then find a basis for the eigenspace.

If $T(A) = A$, then $T(A) = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow a = 0$, $d = 0$, $b = -c$

Matrices $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ Basis $= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

d. (*4 pts*) Without finding the matrix associated with T, find the matrices in the eigenspace associated with $\lambda = -1$, then find a basis for the eigenspace.

If $T(A) = -A$, then $T(A) = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ any a, any d, $b = c$

Matrices $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$

Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

Bonus Question (2 Extra Credit Points):

The **left principal minors** of a matrix are the determinants of the matrices formed by deleting the last row and the last column of a matrix, then the last two columns and last two rows of the original matrix, then the last three rows and last three columns of the original matrix until the matrix is just 1×1.
A matrix is called positive definite of the determinant of the matrix and all the principal minors are positive.
Determine whether A is positive definite.

$$A = \begin{bmatrix} 6 & 1 & 0 & -1 \\ 2 & 2 & 0 & 1 \\ 0 & 3 & 8 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix}$$

$$|A| = 8 \cdot \begin{vmatrix} 6 & 1 & -1 \\ 2 & 2 & 1 \\ 0 & 1 & 5 \end{vmatrix} = 8 \cdot \left(-1 \left(\begin{vmatrix} 6 & -1 \\ 2 & 1 \end{vmatrix} \right) + 5 \left(\begin{vmatrix} 6 & 1 \\ 2 & 2 \end{vmatrix} \right) \right)$$

$$= 8 \cdot \left[(-1)(8) + 5(10) \right] = 8 \cdot 42 = 336 > 0 \checkmark$$

$$\begin{vmatrix} 6 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & -3 & 8 \end{vmatrix} = 8 \begin{vmatrix} 6 & 1 \\ 2 & 2 \end{vmatrix} = 8 \cdot 10 = 80 > 0 \checkmark$$

$$\begin{vmatrix} 6 & 1 \\ 2 & 2 \end{vmatrix} = 10 > 0 \checkmark$$

$6 > 0 \checkmark$

Yes, A is positive definite