1. (*15 points*) Given $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{bmatrix}$. Find a **basis** for the following.

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. (*5 pts*) Col(A)

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

b. (*5 pts*) Nul(A)

$$= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c. (*5 pts*) Row(A)

$$= \{ [1\ 2\ 0\ -2\ 1], [0\ 0\ 1\ 3\ 4] \}$$

2. (*9 points*) Prove (verify in generality) that H = span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_p}\}$ where each $\vec{v_i}$ is in V is a subspace of V.

$\vec{h} \in H \Rightarrow \vec{h} = c_1\vec{v_1} + \ldots + c_p\vec{v_p}$ c_i scalar

1) $\vec{0} \in H$, choose $c_i = 0$

2) $(c_1\vec{v_1} + \ldots + c_p\vec{v_p}) + (d_1\vec{v_1} + \ldots + d_p\vec{v_p}) = (c_1 + d_1)\vec{v_1} + \ldots + (c_p + d_p)\vec{v_p} \in H$

closed under vector addition

3) $k(c_1\vec{v_1} + \ldots + c_p\vec{v_p}) = (kc_1)\vec{v_1} + \ldots + (kc_p)\vec{v_p} \in H$

closed under scalar mult.

3. (*12 points*) $A = \begin{bmatrix} 2 & k \\ 0 & 2 \end{bmatrix}$

a. (*3 pts*) What is the det(A)? 4

b. (*3 pts*) For what k is A invertible? all k

c. (*3 pts*) What are the eigenvalues of A? 2, 2

d. (*3 pts*) For what values of k will A be diagonalizable? only $k = 0$

4. (*13 points*) Determine if the matrix A can be diagonalized. If so, find P and D such that $A = PDP^{-1}$. The characteristic equation for A is $(\lambda+1)^2(\lambda-2) = 0$. (Be careful with your arithmetic).

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 0 & -1 & 0 \\ 6 & 6 & 5 \end{bmatrix}$$

$\lambda = -1$

$$\mathrm{Nul}\left(\begin{bmatrix} -3 & -3 & -3 \\ 0 & 0 & 0 \\ 6 & 6 & 6 \end{bmatrix}\right), \quad \begin{bmatrix} -3 & -3 & -3 \\ 0 & 0 & 0 \\ 6 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis eigenspace $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 2$

$$\mathrm{Nul}\left(\begin{bmatrix} -6 & -3 & -3 \\ 0 & -3 & 0 \\ 6 & 6 & 3 \end{bmatrix}\right), \quad \begin{bmatrix} -6 & -3 & -3 \\ 0 & -3 & 0 \\ 6 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & .5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis eigenspace $= \left\{ \begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} -1 & -1 & -.5 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 2 \end{bmatrix}$$

5. (*11 points*)

a. (*5 pts*) Let $[\vec{x}]_\beta = \begin{bmatrix} \ell \\ v \\ \ell \end{bmatrix}$ be the coordinate vector of $\vec{x}$ with respect to the basis, $\beta = \{\vec{O}, \vec{E}, \vec{A}\}$.

Determine $\vec{x}$.

$$\vec{x} = \ell \cdot \vec{O} + v \cdot \vec{E} + \ell \cdot \vec{A}$$

b. (*6 pts*) Let $H = \{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R}\}$. Show H is isomorphic to $\mathbb{R}^3$ by using the fact that the mapping $\vec{x} \mapsto [\vec{x}]_\beta$ is an isomorphism between H and $\mathbb{R}^3$ when you define a basis β of H and $[\vec{x}]_\beta$ in $\mathbb{R}^3$.

define β, basis of A as $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

then $[\vec{x}]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

since $\vec{x} \longmapsto [\vec{x}]_\beta$ is an isomorphism

H is isomorphic to $\mathbb{R}^3$

6. (*13 points*) Let A be an invertible, diagonalizable matrix ($A = PDP^{-1}$) where

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ ad - bc = 1 \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Find the general form for A^{-k} (Remark: A^{-k} is just $(A^{-1})^k$).

$$A^{-1} = PD^{-1}P^{-1}$$

$$A^{-k} = PD^{-k}P^{-1}$$

$$A^{-k} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}^k \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} a\lambda_1^{-k} & b\lambda_2^{-k} \\ c\lambda_1^{-k} & d\lambda_2^{-k} \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad\lambda_1^{-k} - bc\lambda_2^{-k} & -ab\lambda_1^{-k} + ab\lambda_2^{-k} \\ cd\lambda_1^{-k} - cd\lambda_2^{-k} & -bc\lambda_1^{-k} + ad\lambda_2^{-k} \end{bmatrix}$$

7. (*27 points*) Short answer.

a. (*6 pts*) A matrix is called **full rank** when its rank is as large as possible. Let A be an $m \times n$ matrix of full rank. Justify whether or not $A\vec{x} = \vec{b}$ will be consistent if:

- $m = n$

yes, by IMT or Thm. 4 or every row will have a pivot.

- $m > n$

rank=n

no, does not have a pivot in every row.
Col(A) will not span $\mathbb{R}^m$

- $m < n$

yes, every row has a pivot.
Not unique, but consistent.

b. (*4 pts*) Give an example of a subset H of some vector space V of your choosing that is NOT a subspace. Briefly explain why.

$H \subset \mathbb{R}^2$ $\quad H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in QI} \right\}$ $\quad$ H is not closed under scalar multiplication.

c. (*4 pts*) A is a 6×6 matrix with 4 eigenvalues. Two of the eigenspaces are two-dimensional. Is A always diagonalizable? Why or why not?

Yes, the other 2 must be at least one dimensional. 2 dim + 2 dim + 1 dim + 1 dim = 6

A is always diagonalizable.

d. (*5 pts*) Suppose A is an $n \times n$ matrix. If rank(A) = n, what is the Col(A), Row(A), and the Nul(A).

$\text{Col}(A) = \mathbb{R}^n$

$\text{Row}(A) = \mathbb{R}^n$

$\text{Nul}(A) = \{0\}$

e. (*4 pts*) Let A and B be two matrices where AB exists. Explain why Nul(B) is a subset of Nul(AB) (ie. explain why every vector in Nul(B) is also in Nul(AB).

$\vec{x} \in \text{Nul}(B)$ then $B\vec{x} = \vec{0}$

So $AB\vec{x} = A \cdot \vec{0} = \vec{0}$ so $\vec{x} \in \text{Nul}(AB)$ by def. of nullspace.

f. (*4 pts*) Consider the linear transformation which sends a 2×2 matrix to its transpose. Without making the matrix of this transformation find an eigenvalue and describe the associated eigenspace.

$\lambda = 1$, matrices fixed by this transformation (the eigenspace of $\lambda = 1$) would be matrices of the form $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$A = A^T$.

8. (*1 point*) Bonus: In problem 6, if $\lambda_1 = \lambda_2$ what is the general form of A^{-k}?

$$\begin{bmatrix} \lambda^{-k} & 0 \\ 0 & \lambda^{-k} \end{bmatrix}$$