MATH 205A,B LINEAR ALGEBRA - PROF. P. WONG

EXAM II - NOVEMBER 16, 2015

NAME: **Section:**(Circle one) A(8 : 00) B(9 : 30)

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Let
$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.$$

(a)(7 pts) Find the eigenvalues of A.

The characteristic polynomial is
$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = (3-\lambda)(-\lambda) - 2 = \lambda^2 - 3\lambda - 2.$$

The eigenvalues of A are the roots of this polynomial. The quadratic formula yields
$$\lambda = \frac{3 \pm \sqrt{17}}{2}.$$

The eigenvalues are $\lambda_1 = \frac{3+\sqrt{17}}{2}$ and $\lambda_2 = \frac{3-\sqrt{17}}{2}$.

(b)(7 pts) For each of the eigenvalue(s) found in (a), determine the corresponding eigenspaces by giving a basis for each such subspace.

For the eigenvalue λ_1, consider the matrix $A - \lambda_1 I$. Note that
$$A - \lambda_1 I = \begin{bmatrix} 3-\lambda_1 & 2 \\ 1 & -\lambda_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\lambda_1 \\ 3-\lambda_1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{bmatrix}.$$

It follows that the eigenspace corresponding to λ_1 is given by $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1 = \lambda_1 \right\} = \left\{ x_2 \begin{bmatrix} \frac{3+\sqrt{17}}{2} \\ 1 \end{bmatrix} \right\}$. **Similarly the eigenspace corresponding to λ_2 is** $\left\{ x_2 \begin{bmatrix} \frac{3-\sqrt{17}}{2} \\ 1 \end{bmatrix} \right\}$.

(c)(6 pts) Is A diagonalizable? If so, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

From (a), A has two distinct eigenvalues so A is diagonalizable. From the calculations in (b), the matrix whose columns are the eigenvectors will be one such matrix P, i.e.,
$$P = \begin{bmatrix} \frac{3+\sqrt{17}}{2} & \frac{3-\sqrt{17}}{2} \\ 1 & 1 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}.$$

(a)(10 pts) Find a basis for the column space Col$A$ of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the first two columns have pivots. It follows that $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$ **is a basis for** Col$A$**.**

(b)(10 pts) Find a basis for the null space Nul$A$ of A.

First Nul$A = \{\vec{x} \mid A\vec{x} = \vec{0}\}$**.**

Note that $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$**, it follows that** $x_1 + x_3 + x_4 = 0$ **and** $x_2 + x_3 - x_4 = 0$ **so that** $x_1 = -x_3 - x_4$ **and** $x_2 = -x_3 + x_4$**. Equivalently,**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ **is a basis for** Nul$A$**.**

3. (a)(5 pts) Let A be a 3×4 matrix. If $\dim \mathrm{Nul} A = 3$, what is the rank of A? Justify your answer.

The Rank Theorem says that

$$\dim \mathrm{Nul} A + \mathrm{rank} A = 4.$$

It follows that $\mathrm{rank} A = 1$**.**

(b)(5 pts) Suppose B is a 4×4 matrix with eigenvalues $2, 3, -1$ such that the eigenspace corresponding to 2 has dimension 1; the eigenspace corresponding to 3 has dimension 1; and the eigenspace corresponding to -1 has dimension 1. Determine whether B is diagonalizable. Justify your answer.

Since B has 3 eigenvalues each of which has an eigenspace of dimension, B has 3 linearly independent eigenvectors. However, B is a 4×4 matrix, it follows that B is NOT diagonalizable.

(c)(5 pts) Let B be the matrix as in (b). Find $\det(B + I)$, the determinant of the matrix $(B + I)$. Justify your answer.

Since $\lambda = -1$ is an eigenvalue of B, $\det(B + I) = \det(B - (-1)I) = 0$**.**

(d)(5 pts) Let B be the matrix as in (b). What is the dimension of $\mathrm{Col}(B - 3I)$? Justify your answer.

By the Rank Theorem, $\dim \mathrm{Col}(B - 3I) + \dim \mathrm{Nul}(B - 3I) = 4$**. Note that the null space** $\mathrm{Nul}(B - 3I)$ **is the same as the eigenspace corresponding to the eigenvalue** $\lambda = 3$ **such that it has one dimension. It follows that** $\dim \mathrm{Col}(B - 3I) = 3$**.**

4. (a) Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{R}^3$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ -8 \\ 9 \end{bmatrix}.$$

(i)(4 pts) Find the coordinate matrix $P_{\mathcal{B}}$.

The coordinate matrix is given by

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix}.$$

(ii)(8 pts) Suppose $\vec{u} = \begin{bmatrix} 5 \\ -12 \\ 3 \end{bmatrix}$. Find $[\vec{u}]_{\mathcal{B}}$, the $\mathcal{B}$-coordinates of $\vec{u}$.

First, note that $\vec{u} = P_{\mathcal{B}}[\vec{u}]_{\mathcal{B}}$. Consider the following augmented matrix

$$\begin{bmatrix} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{bmatrix}.$$

A straightforward calculation shows that

$$\begin{bmatrix} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It follows that $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$**.**

(b)(8 pts) **Use row reduction** to find the determinant $\det A$ of the following matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}.$$

$$\det \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & -10 & 3 \\ 2 & 1 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & -10 & 3 \\ 0 & -5 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & -10 & 3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

so $\det A = -5$**.**

5. (a)(7 pts) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be given by $T(\mathbf{p}(x)) = x\mathbf{p}(x)$. Show that T is a linear transformation.

To show that T is a linear transformation, we must show that (i) $T(\mathbf{p}(x) + \mathbf{q}(x)) = T(\mathbf{p}(x)) + T(\mathbf{q}(x))$ and (ii) $T(c\mathbf{p}(x)) = cT(\mathbf{p}(x))$.

By definition of T, we have $T(\mathbf{p}(x)+\mathbf{q}(x)) = x[\mathbf{p}(x)+\mathbf{q}(x)]$ which is equal to $x\mathbf{p}(x)+x\mathbf{q}(x) = T(\mathbf{p}(x)) + T(\mathbf{q}(x))$. Thus (i) holds. Similarly, $T(c\mathbf{p}(x)) = x[c\mathbf{p}(x)] = cx\mathbf{p}(x) = cT(\mathbf{p}(x))$ so that (ii) holds as well. Hence, T is a linear transformation.

(b)(3 pts) Find $KerT$, the kernel of T.

By definition, the kernel $KerT$ is the set of vectors that are mapped to zero under T, that is, $KerT = \{\mathbf{p}(x) \mid T(\mathbf{p}(x)) = \mathbf{0}\}$. If $\mathbf{p}(x) = ax^2 + bx + c$ lies in $KerT$ then $x\mathbf{p}(x) = ax^3 + bx^2 + cx$ must be the zero *polynomial* in $\mathbb{P}_3$. It follows that $a = b = c = 0$. Hence $\mathbf{p}(x)$ must be the zero polynomial in $\mathbb{P}_2$. In other words, $KerT = \{\mathbf{0}\}$.

(c)(5 pts) Let $\mathbf{p}_1(x) = x + x^3$ be in $\mathbb{P}_3$. Does $\mathbf{p}_1(x)$ lie in the Range of T? Justify your answer.

For $\mathbf{p}_1(x)$ to be in the Range of T, it suffices to find some $\mathbf{p}(x) \in \mathbb{P}_2$ such that $T(\mathbf{p}(x)) = \mathbf{p}_1(x)$. Since $T(\mathbf{p}(x)) = x\mathbf{p}(x)$, it is easy to see that $T(1 + x^2) = \mathbf{p}_1(x)$. Since $1 + x^2 \in \mathbb{P}_2$, we conclude that $\mathbf{p}_1(x)$ lies in the Range of T.

(d)(5 pts) Let $\mathbf{p}_2(x) = 1 + x$ be in $\mathbb{P}_3$. Does $\mathbf{p}_2(x)$ lie in the Range of T? Justify your answer.

Similar to (c), we see that if $T(\mathbf{q}(x)) = \mathbf{p}_2(x)$ then $\mathbf{q}(x) = \frac{1}{x}+1$ which is NOT a polynomial in $\mathbb{P}_2$. Equivalently, every element in the Range of T must be a polynomial that has no constant term and $1 + x$ has a nonzero constant term. It follows that $\mathbf{p}_2(x)$ does NOT lie in the Range of T.