Answer Key for Exam #2

1. Use elimination on an augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 & & 7 \\ 1 & 3 & 1 & 3 & 1 & & 8 \\ 2 & 2 & 1 & 2 & 5 & & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & & 7 \\ 0 & 1 & 0 & 2 & 0 & & 1 \\ 0 & -2 & -1 & 0 & 3 & & -5 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & -3 & 1 & & 5 \\ 0 & 1 & 0 & 2 & 0 & & 1 \\ 0 & 0 & -1 & 4 & 3 & & -3 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 4 & & 2 \\ 0 & 1 & 0 & 2 & 0 & & 1 \\ 0 & 0 & 1 & -4 & -3 & & 3 \end{pmatrix}$$

The fourth and fifth columns lack pivots, so x_4 and x_5 are the free variables. The corresponding system is

$$x_1 + x_4 + 4x_5 = 2, \qquad x_2 + 2x_4 = 1, \qquad x_3 - 4x_4 - 3x_5 = 3$$

which we solve for the pivot variables:

$$\begin{array}{lcrrr} x_1 & = & 2 & -x_4 & -4x_5 \\ x_2 & = & 1 & -2x_4 & \\ x_3 & = & 3 & +4x_4 & +3x_5 \\ x_4 & = & & x_4 & \\ x_5 & = & & & x_5 \end{array}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -2 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

2. We perform the eliminations

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 & 1 \\ 1 & 3 & 5 & 9 & 5 \\ -1 & 5 & 3 & 7 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 & 5 & 1 \\ 0 & 2 & 2 & 4 & 4 \\ 0 & 6 & 6 & 12 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 3 & -1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

A basis for the row space is the pivot rows of R, or of A. A basis for the column space is the pivot columns of A (but not of R). A basis for the nullspace can be found as in problem 1 or by taking the negative of the upper right corner

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

of R, putting a 3×3 identity matrix below it, and taking the three columns of that. So the only basis that requires more work is the left nullspace. To get that we transpose the pivot columns of A and eliminate:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \end{pmatrix}$$

Here we can solve the corresponding system, or throw away the 2×2 identity on the left, negate the rest, and put a 1×1 identity under it. We also have another basis for the column space in the rows of the last matrix above. In conclusion

A **row space basis** is $\begin{pmatrix}1\\0\\2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}$ or $\begin{pmatrix}1\\1\\3\\5\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\5\\9\\5\end{pmatrix}$

A **null space basis** is $\begin{pmatrix}-2\\-1\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}-3\\-2\\0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\0\\0\\1\end{pmatrix}$

A **column space basis** is $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\5\end{pmatrix}$ or $\begin{pmatrix}1\\0\\-4\end{pmatrix}$ and $\begin{pmatrix}0\\1\\3\end{pmatrix}$

A **left null space basis** is $\begin{pmatrix}4\\-3\\1\end{pmatrix}$

The factored form of A that displays bases for all four is

$$A = \begin{pmatrix}1 & 0\\0 & 1\\-4 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 2 & 3 & -1\\0 & 1 & 1 & 2 & 2\end{pmatrix}$$

3. Taking the easy one first, the left nullspace is one-dimensional, so all we have to do is divide by the length, which is $\sqrt{4^2+(-3)^2+1^2} = \sqrt{26}$. So

An **orthonormal** left null space basis is $\frac{1}{\sqrt{26}}\begin{pmatrix}4\\-3\\1\end{pmatrix}$

Our nicest column space vector is probably $\vec{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, so we keep it and change $\vec{b} = \begin{pmatrix}1\\3\\5\end{pmatrix}$. We have $\vec{a}\cdot\vec{a} = 1+1+1 = 3$ and $\vec{a}\cdot\vec{b} = 1+3-5 = -1$, so the projection of $\vec{b}$ onto $\vec{a}$ is

$$\frac{\vec{a}\cdot\vec{a}}{\vec{a}\cdot\vec{b}}\vec{a} = \frac{-1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}1\\3\\5\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection, which is what we want to replace $\vec{b}$ with. We have

$$\vec{e} = \begin{pmatrix}1\\3\\5\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} = \frac{1}{3}\left\{\begin{pmatrix}3\\9\\15\end{pmatrix} + \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\} = \frac{1}{3}\begin{pmatrix}4\\10\\14\end{pmatrix} = \frac{2}{3}\begin{pmatrix}2\\5\\7\end{pmatrix}$$

and we can get rid of the factor $\frac{2}{3}$. So

An **orthogonal** column space basis is $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\5\\7\end{pmatrix}$

and therefore

An **orthonormal** column space basis is $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\frac{1}{\sqrt{78}}\begin{pmatrix}2\\5\\7\end{pmatrix}$

since $1^2+1^2+(-1)^2=3$ and $2^2+5^2+7^2=78$. Our nicest row space vector appears to be $\vec{a}=\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}$, so

we keep it and replace $\vec{b}=\begin{pmatrix}1\\0\\2\\3\\-1\end{pmatrix}$. We have $\vec{a}\cdot\vec{a}=0+1+1+4+4=10$ and $\vec{a}\cdot\vec{b}=0+0+2+6-2=6$,
so the projection of $\vec{b}$ onto $\vec{a}$ is

$$\frac{\vec{a}\cdot\vec{a}}{\vec{a}\cdot\vec{b}}\vec{a}=\frac{6}{10}\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}$$

Therefore

$$\begin{pmatrix}1\\0\\2\\3\\-1\end{pmatrix}=\frac{3}{5}\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}+\vec{e}$$

where $\vec{e}$ is the error in the projection, which we want to replace $\vec{b}$ with. We have

$$\vec{e}=\begin{pmatrix}1\\0\\2\\3\\-1\end{pmatrix}-\frac{3}{5}\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}=\frac{1}{5}\left\{\begin{pmatrix}5\\0\\10\\15\\-5\end{pmatrix}-\begin{pmatrix}0\\3\\3\\6\\6\end{pmatrix}\right\}=\frac{1}{5}\begin{pmatrix}5\\-3\\7\\9\\-11\end{pmatrix}$$

and we can get rid of the constant factor $\frac{1}{5}$. So

An **orthogonal** row space basis is $\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}5\\-3\\7\\9\\-11\end{pmatrix}$

and therefore

An **orthonormal** row space basis is $\frac{1}{\sqrt{10}}\begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix}$ and $\frac{1}{\sqrt{285}}\begin{pmatrix}5\\-3\\7\\9\\-11\end{pmatrix}$

since $0^2+1^2+1^2+2^2+2^2=10$ and $5^2+(-3)^2+7^2+9^2+(-11)^2=285$.
Finally we have to get an orthonormal basis for the null space of A. We have a small bit of luck in that two of our null space basis vectors are already perpendicular, namely $\begin{pmatrix}-2\\-1\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\0\\0\\1\end{pmatrix}$, so we just have

to project $\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ onto them. We can do these projections one at a time by the same method as above, or both at once by finding the projection matrix onto the other two. We have

$$\begin{pmatrix} -2 & -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & -2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 6I,$$

so the projection matrix is

$$P = \frac{1}{6} \begin{pmatrix} -2 & 1 \\ -1 & -2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & 0 & -2 & 0 & 1 \\ 0 & 5 & -1 & 0 & -2 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix}$$

Applying this to $\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ we get

$$\frac{1}{6} \begin{pmatrix} 5 & 0 & -2 & 0 & 1 \\ 0 & 5 & -1 & 0 & -2 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -15 \\ -10 \\ 8 \\ 0 \\ 1 \end{pmatrix}$$

The error $\vec{e}$ in the projection satisfies $\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -15 \\ -10 \\ 8 \\ 0 \\ 1 \end{pmatrix} + \vec{e}$ so

$$\vec{e} = \frac{1}{6} \begin{pmatrix} -18 \\ -12 \\ 0 \\ 6 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -15 \\ -10 \\ 8 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 \\ -2 \\ -8 \\ 6 \\ -1 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 3 \\ 2 \\ 8 \\ -6 \\ 1 \end{pmatrix}.$$

So we replace $\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ by $\begin{pmatrix} 3 \\ 2 \\ 8 \\ -6 \\ 1 \end{pmatrix}$. Then

An **orthogonal** null space basis is $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 8 \\ -6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The lengths here are $\sqrt{4+1+1+0+0} = \sqrt{6}$, $\sqrt{9+4+64+36+1} = \sqrt{114}$, and $\sqrt{1+4+0+0+1} = \sqrt{6}$, respectively. Therefore

An **orthonormal** null space basis is $\frac{1}{\sqrt{6}}\begin{pmatrix}-2\\-1\\1\\0\\0\end{pmatrix}$ and $\frac{1}{\sqrt{114}}\begin{pmatrix}3\\2\\8\\-6\\1\end{pmatrix}$ and $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\0\\0\\1\end{pmatrix}$

4. Let A be the matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns: $A = \begin{pmatrix}3 & 1\\1 & -1\\1 & 3\\1 & -1\\-2 & 2\end{pmatrix}$. Then

$$A^TA = \begin{pmatrix}3 & 1 & 1 & 1 & -2\\1 & -1 & 3 & -1 & 2\end{pmatrix}\begin{pmatrix}3 & 1\\1 & -1\\1 & 3\\1 & -1\\-2 & 2\end{pmatrix} = \begin{pmatrix}16 & 0\\0 & 16\end{pmatrix} = 16I,$$

so the inverse is $\frac{1}{16}I$, and therefore

$$P = \frac{1}{16}AA^T = \frac{1}{16}\begin{pmatrix}3 & 1\\1 & -1\\1 & 3\\1 & -1\\-2 & 2\end{pmatrix}\begin{pmatrix}3 & 1 & 1 & 1 & -2\\1 & -1 & 3 & -1 & 2\end{pmatrix} = \frac{1}{16}\begin{pmatrix}10 & 2 & 6 & 2 & -4\\2 & 2 & -2 & 2 & -4\\6 & -2 & 10 & -2 & 4\\2 & 2 & -2 & 2 & -4\\-4 & -4 & 4 & -4 & 8\end{pmatrix},$$

so the projection matrix P onto the subspace S is $P = \frac{1}{8}\begin{pmatrix}5 & 1 & 3 & 1 & -2\\1 & 1 & -1 & 1 & -2\\3 & -1 & 5 & -1 & 2\\1 & 1 & -1 & 1 & -2\\-2 & -2 & 2 & -2 & 4\end{pmatrix}$. We also have

$$R = 2P - I = \frac{1}{4}\begin{pmatrix}5 & 1 & 3 & 1 & -2\\1 & 1 & -1 & 1 & -2\\3 & -1 & 5 & -1 & 2\\1 & 1 & -1 & 1 & -2\\-2 & -2 & 2 & -2 & 4\end{pmatrix} - \frac{1}{4}\begin{pmatrix}4 & 0 & 0 & 0 & 0\\0 & 4 & 0 & 0 & 0\\0 & 0 & 4 & 0 & 0\\0 & 0 & 0 & 4 & 0\\0 & 0 & 0 & 0 & 4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1 & 3 & 1 & -2\\1 & -3 & -1 & 1 & -2\\3 & -1 & 1 & -1 & 2\\1 & 1 & -1 & -3 & -2\\-2 & -2 & 2 & -2 & 0\end{pmatrix}$$

is the reflection matrix through S. The projection $\vec{p}$ of $\vec{v}_3 = \begin{pmatrix}3\\2\\1\\2\\3\end{pmatrix}$ onto S is

$$\vec{p} = P\vec{v}_3 = \frac{1}{8}\begin{pmatrix}5 & 1 & 3 & 1 & -2\\1 & 1 & -1 & 1 & -2\\3 & -1 & 5 & -1 & 2\\1 & 1 & -1 & 1 & -2\\-2 & -2 & 2 & -2 & 4\end{pmatrix}\begin{pmatrix}3\\2\\1\\2\\3\end{pmatrix} = \frac{1}{8}\begin{pmatrix}16\\0\\16\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\2\\0\\0\end{pmatrix},$$

and the reflection $\vec{r}$ of $\vec{v}_3$ through S is

$$\vec{r} = R\vec{v}_3 = \frac{1}{4}\begin{pmatrix}1 & 1 & 3 & 1 & -2\\1 & -3 & -1 & 1 & -2\\3 & -1 & 1 & -1 & 2\\1 & 1 & -1 & -3 & -2\\-2 & -2 & 2 & -2 & 0\end{pmatrix}\begin{pmatrix}3\\2\\1\\2\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4\\-8\\12\\-8\\-12\end{pmatrix} = \begin{pmatrix}1\\-2\\3\\-2\\-3\end{pmatrix}.$$

There are (at least) two different ways of describing the relationship between $\vec{v}_3$, $\vec{p}$ and $\vec{r}$. We can compute the error $\vec{e}$ in the projection:

$$\vec{v}_3 = \vec{p} + \vec{e}, \quad \text{so} \quad \vec{e} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 3 \end{pmatrix}.$$

Then we should have $\vec{r} = \vec{p} - \vec{e}$, which we can check:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 3 \end{pmatrix} \qquad \text{which is indeed correct.}$$

The relationship between $\vec{v}_3$, $\vec{p}$ and $\vec{r}$ may be described either as this pair of equations ($\vec{v}_3 = \vec{p} + \vec{e}$ and $\vec{r} = \vec{p} - \vec{e}$), or as the single equation which comes from adding them together: $\vec{v}_3 + \vec{r} = 2\vec{p}$. $\vec{p}$ is the average of $\vec{v}_3$ and $\vec{r}$.

5. If $P = \frac{1}{10}\begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}$ then P is symmetric and

$$P^2 = \frac{1}{100}\begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 70 & 40 & 10 & -20 \\ 40 & 30 & 20 & 10 \\ 10 & 20 & 30 & 40 \\ -20 & 10 & 40 & 70 \end{pmatrix} = P.$$

Any matrix P which satisfies $P^2 = P = P^T$ is a projection matrix. There is a cheap way of doing the rest of the problem: the trace of P is $\frac{1}{10}(7+3+3+7) = 2$, so the subspace S that P projects onto is 2-dimensional. Therefore any two rows or columns of P will be a basis for it as long as they are not multiples of each other. Thus, for example, $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ are a basis for S. A basis for $S^\perp$ can be found as the nullspace of these two, or by writing down $I - P$ and using the same trick:

$$I - P = \frac{1}{10}\begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{pmatrix}.$$

This projects onto $S^\perp$, which is also 2-dimensional, so again any 2 rows or columns of $I - P$ are a basis for $S^\perp$ as long as they're not multiples of each other.

Alternatively, we could eliminate on P to find the two bases we want:

$$P = \frac{1}{10}\begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -10 & -20 & -30 \\ 0 & -5 & -10 & -15 \\ 1 & 2 & 3 & 4 \\ 0 & 5 & 10 & 15 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

P projects onto its own column space, or its row space since P is symmetric. This gives us a nice basis for S, and a basis for $S^{\perp}$ comes from the null space of P:

$$\textbf{A basis for } S \text{ is} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \textbf{A basis for } S^{\perp} \text{ is} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

6. Q is an orthogonal matrix if $Q^{-1} = Q^T$. This certainly holds for the identity matrix I, since $I^T = I = I^{-1}$. If Q_1 and Q_2 are orthogonal then $Q_1^{-1} = Q_1^T$ and $Q_2^{-1} = Q_2^T$, so

$$\begin{aligned} (Q_1 Q_2)^{-1} &= Q_2^{-1} Q_1^{-1} && \text{the inverse distributes, with the order changed} \\ &= Q_2^T Q_1^T && \text{since } Q_1 \text{ and } Q_2 \text{ are orthogonal} \\ &= (Q_1 Q_2)^T && \text{the transpose distributes, with the order changed} \end{aligned}$$

Therefore $Q_1 Q_2$ is an orthogonal matrix. Finally, if Q is an orthogonal matrix then

$$\begin{aligned} \left(Q^{-1}\right)^T &= \left(Q^T\right)^{-1} && \text{inverting and transposing commute} \\ &= \left(Q^{-1}\right)^{-1} && \text{since } Q \text{ is orthogonal,} \end{aligned}$$

and therefore Q^{-1} is orthogonal. Therefore the set of all $n \times n$ orthogonal matrices is a group, for each positive integer n.