Exam #2, Math 205B (Linear Algebra)

This take-home exam is due at class time on **Friday, November 16**. You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the bottom of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Please show all work.

1. (16 points) Find the complete solution of the system $\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 & 1 \\ 2 & 2 & 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

2. (22 points) Find a basis for each of the four subspaces associated with the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 & 1 \\ 1 & 3 & 5 & 9 & 5 \\ -1 & 5 & 3 & 7 & 11 \end{pmatrix}$$

What is the factored form of A that displays these bases?

3. (16 points) Find an **orthonormal** basis for each of the four subspaces in problem 2.

4. (22 points) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, and let S be the subspace with basis $\{\vec{v}_1, \vec{v}_2\}$. Find the matrix P that projects vectors in $\mathbb{R}^5$ onto S, and the matrix R that reflects through S. Find also the projection $\vec{p}$ of $\vec{v}_3$ onto S and the reflection $\vec{r}$ of $\vec{v}_3$ through S. How are $\vec{v}_3$, $\vec{p}$ and $\vec{r}$ related?

5. (14 points) Explain how you can tell that $P = \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}$ is a projection matrix. Find a basis for the subspace S of $\mathbb{R}^4$ that P projects onto, and a basis for $S^\perp$ (the orthogonal complement of S).

6. (10 points) In Math 309 one studies certain kinds of algebraic structures, of which perhaps the most important are **groups**. It turns out that every group can be described in terms of matrices. In this problem I want you to show that the set of all $n \times n$ orthogonal matrices is a group. This entails 4 things: closure, associativity, identity and inverses. Since we are working with matrices the associativity is automatic, so you have to check three things:

(i) Is the $n \times n$ identity matrix an orthogonal matrix? (identity)
(ii) If Q_1 and Q_2 are orthogonal matrices, is $Q_1 Q_2$ an orthogonal matrix? (closure)
(iii) If Q is an orthogonal matrix, is Q^{-1} an orthogonal matrix? (inverses)

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ____________________