Answer Key for Exam #2

1. Use elimination on an augmented matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 & 2 & 10 \\ 3 & 2 & -11 & 1 & 8 & 25 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 3 \\ 0 & -1 & -5 & -1 & 0 & 4 \\ 0 & -4 & -20 & -5 & 5 & 16 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -7 & 0 & 1 & 11 \\ 0 & 1 & 5 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 5 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -7 & 0 & 1 & 11 \\ 0 & 1 & 5 & 0 & 5 & -4 \\ 0 & 0 & 0 & 1 & -5 & 0 \end{pmatrix}.$$

The corresponding system is

$$x_1 - 7x_3 + x_5 = 11, \qquad x_2 + 5x_3 + 5x_5 = -4, \qquad x_4 - 5x_5 = 0$$

which we solve for the pivot variables x_1, x_2 and x_4:

$$\begin{aligned} x_1 &= 11 + 7x_3 - x_5 \\ x_2 &= -4 - 5x_3 - 5x_5 \\ x_3 &= x_3 \\ x_4 &= 5x_5 \\ x_5 &= x_5 \end{aligned}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -5 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

2. We perform the eliminations

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & -4 & 7 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -5 & 5 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

A basis for the row space is the pivot rows of R, or of A. A basis for the column space is the pivot columns of A (but not of R). A basis for the nullspace can be found as in problem 1 or by taking the negative of the upper right corner

$$\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}$$

of R, putting a 2×2 identity matrix below it, and taking the two columns of that. So the only basis that requires more work is the left nullspace. To get it we transpose the pivot columns of A and eliminate:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -5 \end{pmatrix}.$$

Here we can solve the corresponding system, or throw away the 2×2 identity on the left, negate the rest, and put a 1×1 identity under it. We also have another basis for the column space in the rows of the last matrix above. In conclusion

A **row space basis** is $\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

A **null space basis** is
$$\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

A **column space basis** is
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$$

A **left null space basis** is
$$\begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix}$$

The factored form of A that displays bases for all four is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 6 & -5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

3. To see which vector to keep we start by computing all the dot products for the three vectors. If

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 4 \\ -1 \\ 5 \\ -2 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

then

$$\begin{aligned} &\vec{v}_1 \cdot \vec{v}_2 = 4 - 2 + 5 - 2 = 5, && \vec{v}_1 \cdot \vec{v}_3 = 2 + 2 + 3 + 2 = 9, && \vec{v}_2 \cdot \vec{v}_3 = 8 - 1 + 15 - 4 = 18, \\ &\vec{v}_1 \cdot \vec{v}_1 = 1 + 4 + 1 + 1 = 7, && \vec{v}_2 \cdot \vec{v}_2 = 16 + 1 + 25 + 4 = 46, && \vec{v}_3 \cdot \vec{v}_3 = 4 + 1 + 9 + 4 = 18 \end{aligned}$$

Recall that the projection of $\vec{b}$ onto $\vec{a}$ is $\dfrac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\,\vec{a}$. If we take $\vec{a} = \vec{v}_3$ then the ratios will be $\frac{9}{18}$ and $\frac{18}{18}$, so $\vec{v}_3$ seems like a good one to keep. Then the projection of $\vec{v}_1$ onto $\vec{v}_3$ is

$$\frac{\vec{v}_1 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\,\vec{v}_3 = \frac{9}{18} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix},$$

and therefore

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection. We want to replace $\vec{v}_1$ by some multiple of $\vec{e}$. We have

$$2\vec{e} = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} = \vec{w}_1,$$

so we throw away $\vec{v}_1$ and replace it by $\vec{w}_1$. Next we do the same thing with $\vec{v}_2$. The projection of $\vec{v}_2$ onto $\vec{v}_3$ is

$$\frac{\vec{v}_2 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\,\vec{v}_3 = \frac{18}{18} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix},$$

and therefore

$$\vec{v}_2 = \begin{pmatrix} 4 \\ -1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection, and again we want to replace $\vec{v}_2$ by some multiple of $\vec{e}$. We have

$$\vec{e} = \begin{pmatrix} 4 \\ -1 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} = \vec{w}_2,$$

so we throw away $\vec{v}_2$ and replace it by $\vec{w}_2$. If we rename $\vec{v}_3$ as $\vec{w}_3$, we now have

$$\vec{w}_1 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \vec{w}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix},$$

where $\vec{w}_1 \perp \vec{w}_3$ and $\vec{w}_2 \perp \vec{w}_3$, but we probably don't have $\vec{w}_1 \perp \vec{w}_2$; in fact, $\vec{w}_1 \cdot \vec{w}_2 = 0 - 3 - 1 + 0 = -4 \neq 0$, so we definitely don't have $\vec{w}_1 \perp \vec{w}_2$. To fix this we need to project one of $\vec{w}_1$ and $\vec{w}_2$ onto the other one and replace the projected vector by the error. We have $\vec{w}_1 \cdot \vec{w}_1 = 9 + 1 = 10$ and $\vec{w}_2 \cdot \vec{w}_2 = 1 + 1 + 1 + 4 = 7$, so it looks like keeping $\vec{w}_1$ might be slightly better. The projection of $\vec{w}_2$ onto $\vec{w}_1$ is

$$\frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1}\,\vec{w}_1 = \frac{-4}{10}\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} = -\frac{2}{5}\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix},$$

and therefore

$$\vec{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} = -\frac{2}{5}\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} + \vec{e},$$

and we want to replace $\vec{w}_2$ by a multiple of $\vec{e}$. We have

$$5\vec{e} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \\ -10 \end{pmatrix},$$

which is the third vector we need. We now have that

$$\begin{pmatrix} 5 \\ 1 \\ 3 \\ -10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

are all perpendicular to each other, and we only have to fix the lengths. The dot product of the first vector with itself is 135, and we did the other two earlier, so we finally get that an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ is

$$\frac{1}{\sqrt{135}}\begin{pmatrix} 5 \\ 1 \\ 3 \\ -10 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{10}}\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{18}}\begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}.$$

Some other possible answers are

$$\frac{1}{\sqrt{7}}\begin{pmatrix}1\\-1\\1\\-2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{378}}\begin{pmatrix}4\\17\\-3\\-8\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{18}}\begin{pmatrix}2\\1\\3\\2\end{pmatrix}$$

$$\frac{1}{\sqrt{7}}\begin{pmatrix}1\\2\\1\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{135}}\begin{pmatrix}5\\1\\3\\-10\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{315}}\begin{pmatrix}5\\-11\\12\\5\end{pmatrix}$$

$$\frac{1}{\sqrt{7}}\begin{pmatrix}1\\2\\1\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{99}}\begin{pmatrix}-1\\-5\\3\\8\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2079}}\begin{pmatrix}23\\-17\\30\\-19\end{pmatrix}$$

$$\frac{1}{\sqrt{46}}\begin{pmatrix}4\\-1\\5\\-2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{378}}\begin{pmatrix}4\\17\\-3\\-8\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{1449}}\begin{pmatrix}5\\16\\12\\32\end{pmatrix}$$

$$\frac{1}{\sqrt{46}}\begin{pmatrix}4\\-1\\5\\-2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{99}}\begin{pmatrix}-1\\-5\\3\\8\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{13662}}\begin{pmatrix}26\\97\\21\\56\end{pmatrix}$$

4. Let A be the matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns. Then

$$A^TA = \begin{pmatrix}1 & 3 & 4 & 3\\ 2 & 1 & 2 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 1\\ 4 & 2\\ 3 & -1\end{pmatrix} = \begin{pmatrix}35 & 10\\ 10 & 10\end{pmatrix},$$

and

$$\begin{pmatrix}35 & 10\\ 10 & 10\end{pmatrix}^{-1} = \frac{1}{35\cdot 10 - 10\cdot 10}\begin{pmatrix}10 & -10\\ -10 & 35\end{pmatrix} = \frac{1}{50}\begin{pmatrix}2 & -2\\ -2 & 7\end{pmatrix},$$

so

$$P = \frac{1}{50}\begin{pmatrix}1 & 2\\ 3 & 1\\ 4 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}2 & -2\\ -2 & 7\end{pmatrix}\begin{pmatrix}1 & 3 & 4 & 3\\ 2 & 1 & 2 & -1\end{pmatrix} = \frac{1}{50}\begin{pmatrix}-2 & 12\\ 4 & 1\\ 4 & 6\\ 8 & -13\end{pmatrix}\begin{pmatrix}1 & 3 & 4 & 3\\ 2 & 1 & 2 & -1\end{pmatrix},$$

and so we find that the projection matrix P onto the subspace S is

$$P = \frac{1}{50}\begin{pmatrix}22 & 6 & 16 & -18\\ 6 & 13 & 18 & 11\\ 16 & 18 & 28 & 6\\ -18 & 11 & 6 & 37\end{pmatrix}.$$

We also have that

$$R = 2P - I = \frac{1}{25}\begin{pmatrix}22 & 6 & 16 & -18\\ 6 & 13 & 18 & 11\\ 16 & 18 & 28 & 6\\ -18 & 11 & 6 & 37\end{pmatrix} - \frac{1}{25}\begin{pmatrix}25 & 0 & 0 & 0\\ 0 & 25 & 0 & 0\\ 0 & 0 & 25 & 0\\ 0 & 0 & 0 & 25\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-3 & 6 & 16 & -18\\ 6 & -12 & 18 & 11\\ 16 & 18 & 3 & 6\\ -18 & 11 & 6 & 12\end{pmatrix}$$

is the reflection matrix through S. The projection of $\vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ onto S is

$$P\vec{v}_3 = \frac{1}{50}\begin{pmatrix} 22 & 6 & 16 & -18 \\ 6 & 13 & 18 & 11 \\ 16 & 18 & 28 & 6 \\ -18 & 11 & 6 & 37 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} -50 \\ 100 \\ 100 \\ 200 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 4 \end{pmatrix},$$

and the reflection of $\vec{v}_3$ through S is

$$R\vec{v}_3 = \frac{1}{25}\begin{pmatrix} -3 & 6 & 16 & -18 \\ 6 & -12 & 18 & 11 \\ 16 & 18 & 3 & 6 \\ -18 & 11 & 6 & 12 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -75 \\ 25 \\ 100 \\ 75 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

The projection is the average of the reflection and $\vec{v}_3$ itself, and this could have been used to avoid one of the last two matrix multiplications.

5. If $P = \frac{1}{31}\begin{pmatrix} 2 & 5 & 2 & 2 & 5 \\ 5 & 28 & 5 & 5 & -3 \\ 2 & 5 & 2 & 2 & 5 \\ 2 & 5 & 2 & 2 & 5 \\ 5 & -3 & 5 & 5 & 28 \end{pmatrix}$ then P is symmetric and

$$P^2 = \frac{1}{961}\begin{pmatrix} 62 & 310 & 62 & 62 & 310 \\ 310 & 868 & 310 & 310 & -93 \\ 62 & 310 & 62 & 62 & 310 \\ 62 & 310 & 62 & 62 & 310 \\ 310 & -93 & 310 & 310 & 868 \end{pmatrix} = P.$$

Any matrix P for which $P^2 = P = P^T$ is a projection matrix. The trace of P is $\frac{1}{31}(2+28+2+2+28) = 2$, so the subspace T that P projects onto is 2-dimensional. Therefore any two rows or columns of P will be a basis for it as long as they are not multiples of each other. But since we also have to find a basis for $T^\perp$, which must be 3-dimensional, let's eliminate:

$$P = \frac{1}{31}\begin{pmatrix} 2 & 5 & 2 & 2 & 5 \\ 5 & 28 & 5 & 5 & -3 \\ 2 & 5 & 2 & 2 & 5 \\ 2 & 5 & 2 & 2 & 5 \\ 5 & -3 & 5 & 5 & 28 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 5 & 2 & 2 & 5 \\ 1 & 18 & 1 & 1 & -13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -31 & 0 & 0 & 31 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & -31 & 0 & 0 & 31 \\ 1 & 18 & 1 & 1 & -13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

P projects onto its own column space, or its row space since P is symmetric. This gives us a nice basis for T, and a basis for $T^\perp$ comes from the null space of P:

A basis for T is $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and **A basis for $T^\perp$ is** $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

6. We have the three conditions

(i) $P^T = P$ (P is symmetric) (ii) $P^2 = P$ and (iii) $P^T P = P$,

and we have to show that (i) and (ii) together imply (iii), and that (iii) implies both (i) and (ii). That (i) and (ii) together imply (iii) is easy: $P^T P = PP$ by (i), which equals P by (ii). The trickiest part is to show (iii) implies (i). To see this transpose both sides of (iii) to get $\left(P^T P\right)^T = P^T$. But

$$\left(P^T P\right)^T = P^T \left(P^T\right)^T = P^T P, \qquad \text{so (iii) implies} \qquad P = P^T P = \left(P^T P\right)^T = P^T,$$

which proves (i). So if (iii) holds then (i) does, and then we can get (ii) easily from (i) and (iii): $PP = P^T P$ by (i), which equals P by (iii).

7. If M is a 2×2 matrix which satisfies $M^2 = M^T$, then M would be a projection matrix by the result used in problems 5 and 6 if $M^2 = M = M^T$. In other words, if $M^2 = M^T$ and M is also symmetric, then M must be a projection matrix. Therefore, the question boils down to: are there any 2×2 matrices M which satisfy $M^2 = M^T$ but are not symmetric? It turns out that there are exactly two such matrices.

$$\text{If} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad M^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

This gives us the four equations

(i) $a^2 + bc = a$, (ii) $b(a+d) = c$, (iii) $c(a+d) = b$, (iv) $d^2 + bc = d$.

Subtracting (iv) from (i) and (iii) from (ii) we get

(v) $a^2 - d^2 = a - d$ and (vi) $(b-c)(a+d) = c - b$.

(vi) implies that either $b = c$ or $a + d = -1$. If $b = c$ then M is symmetric, so to get a nonsymmetric M we must have $a + d = -1$, in which case (ii) and (iii) imply that $b = -c$. (v) implies that either $a = d$ or $a + d = 1$, and the latter is impossible since we already have $a + d = -1$. So $a = d = -\frac{1}{2}$, and since $b = -c$ (i) and (iv) both become

$$\frac{1}{4} + b(-b) = -\frac{1}{2}, \quad \text{or} \quad b^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Therefore either $b = \dfrac{\sqrt{3}}{2} = -c$, or $c = \dfrac{\sqrt{3}}{2} = -b$. Thus the two 2×2 M's which satisfy $M^2 = M^T$ but are not projection matrices are

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

These are clockwise and counterclockwise rotation matrices by 120°, respectively. When they are squared they become 240° rotations, which are 120° rotations in the other direction. So the square equals the inverse for these matrices, and since they are rotations the inverse also equals the transpose.

Scores: The median is 83 and the mean about 83.9, with a high of 96 and a low of 69. Half of the scores were between 82 and 84.