Exam #2, Math 205A (Linear Algebra)

This take-home exam is due on **Friday, November 15** by 5 PM. (Sooner is fine.) You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the bottom of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Matrix multiplications and reduced row echelon forms may be done on MATLAB or a calculator, but please show all other work.

1. (16 points) Find the complete solution of the system $\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 1 & 3 & 2 \\ 3 & 2 & -11 & 1 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 25 \end{pmatrix}$.

2. (20 points) Find a basis for each of the four subspaces associated with the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & -4 & 7 & -4 \end{pmatrix}$$

What is the factored form of A that displays these bases?

3. (14 points) The vectors $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ -1 \\ 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$ span a 3-dimensional subspace of $\mathbb{R}^4$. Find an orthonormal basis for it.

4. (18 points) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$, and let S be the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$. Find the matrix P that projects vectors in $\mathbb{R}^4$ onto S, and the matrix R that reflects through S. Find also the projection $\vec{p}$ of $\vec{v}_3$ onto S and the reflection $\vec{r}$ of $\vec{v}_3$ through S.

5. (16 points) Explain how you can tell that $P = \frac{1}{31}\begin{pmatrix} 2 & 5 & 2 & 2 & 5 \\ 5 & 28 & 5 & 5 & -3 \\ 2 & 5 & 2 & 2 & 5 \\ 2 & 5 & 2 & 2 & 5 \\ 5 & -3 & 5 & 5 & 28 \end{pmatrix}$ is a projection matrix. Find a basis for the subspace T of $\mathbb{R}^5$ that P projects onto, and a basis for $T^\perp$ (the orthogonal complement of T).

6. (10 points) We proved in class that P is a projection matrix if and only if P satisfies the two conditions

(i) $P^T = P$ (P is symmetric) and (ii) $P^2 = P$.

The object of this problem is to reduce these two conditions to the single condition (iii) $P^T P = P$.

Show that if P satisfies (i) and (ii) then it satisfies (iii), and that if P satisfies (iii) then it satisfies (i) and (ii). This would prove that P is a projection matrix if and only if it satisfies (iii).

7. (6 points) Suppose M is a 2×2 matrix which satisfies $M^2 = M^T$. Must M be a projection matrix? If so, explain why. If not, what other possibilities are there for M?

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________