Answer Key for Exam #2

1. Use elimination on an augmented matrix:

$$\begin{pmatrix} 1 & 3 & 1 & 2 & 2 & & 3 \\ 1 & 2 & 1 & 4 & 3 & & 4 \\ 1 & 4 & 2 & 2 & 1 & & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 1 & 2 & 2 & & 3 \\ 0 & -1 & 0 & 2 & 1 & & 1 \\ 0 & 1 & 1 & 0 & -1 & & -1 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 8 & 5 & & 6 \\ 0 & 1 & 0 & -2 & -1 & & -1 \\ 0 & 0 & 1 & 2 & 0 & & 0 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 6 & 5 & & 6 \\ 0 & 1 & 0 & -2 & -1 & & -1 \\ 0 & 0 & 1 & 2 & 0 & & 0 \end{pmatrix}.$$

The corresponding system is

$$x_1 + 6x_4 + 5x_5 = 6, \qquad x_2 - 2x_4 - x_5 = -1, \qquad x_3 + 2x_4 = 0$$

which we solve for the pivot variables x_1, x_2 and x_3:

$$\begin{array}{rcrrr} x_1 & = & 6 & -6x_4 & -5x_5 \\ x_2 & = & -1 & +2x_4 & +x_5 \\ x_3 & = & & -2x_4 & \\ x_4 & = & & x_4 & \\ x_5 & = & & & x_5 \end{array}$$

Therefore

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -6 \\ 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2. We perform the eliminations

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ 2 & 1 & 2 & 7 \\ 2 & 3 & 10 & 13 \\ 5 & 3 & 7 & 19 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 & 5 \\ 0 & -1 & -4 & -3 \\ 0 & 1 & 4 & 3 \\ 0 & -2 & -8 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

A basis for the row space is the pivot rows of R, or of A. A basis for the column space is the pivot columns of A (but not of R). A basis for the nullspace can be found as in problem 1 or by taking the negative of the upper right corner

$$\begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}$$

of R, putting a 2×2 identity matrix below it, and taking the two columns of that. So the only basis that requires more work is the left nullspace. To get it we transpose the pivot columns of A and eliminate:

$$\begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 1 & 3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 5 \\ 0 & -1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix}.$$

Here we can solve the corresponding system, or throw away the 2×2 identity on the left, negate the rest, and put a 2×2 identity under it. We also have another basis for the column space in the rows of the last matrix above. In conclusion

A **row space basis** is $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 2 \\ 7 \end{pmatrix}$

A **null space basis** is $\begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

A **column space basis** is $\begin{pmatrix} 1 \\ 2 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$

A **left null space basis** is $\begin{pmatrix} -4 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$

The factored form of A that displays bases for all four is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 4 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & 3 \end{pmatrix}$$

3. To see which vector to keep we start by computing all the dot products for the three vectors. If

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix}$$

then

$$\begin{aligned} &\vec{v}_1 \cdot \vec{v}_2 = 1 + 2 + 1 + 10 = 14, \quad &&\vec{v}_1 \cdot \vec{v}_3 = 1 + 4 + 1 + 8 = 14, \quad &&\vec{v}_2 \cdot \vec{v}_3 = 1 + 8 + 1 + 20 = 30, \\ &\vec{v}_1 \cdot \vec{v}_1 = 1 + 1 + 1 + 4 = 7, &&\vec{v}_2 \cdot \vec{v}_2 = 1 + 4 + 1 + 25 = 31, &&\vec{v}_3 \cdot \vec{v}_3 = 1 + 16 + 1 + 16 = 34 \end{aligned}$$

Recall that the projection of $\vec{b}$ onto $\vec{a}$ is $\dfrac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\,\vec{a}$. If we take $\vec{a} = \vec{v}_1$ then both ratios will be $\frac{14}{7} = 2$, so $\vec{v}_1$ seems like a good one to keep. Then the projection of $\vec{v}_2$ onto $\vec{v}_1$ is

$$\frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 = 2\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix},$$

and therefore

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix} = 2\vec{v}_1 + \vec{e} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection. We want to replace $\vec{v}_2$ by some multiple of $\vec{e}$. We have

$$\vec{e} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = -\vec{w}_2, \quad \text{where} \quad \vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

so we throw away $\vec{v}_2$ and replace it by $\vec{w}_2$. Next we do the same thing with $\vec{v}_3$. The projection of $\vec{v}_3$ onto $\vec{v}_1$ is

$$\frac{\vec{v}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

again, and therefore

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix} = 2\vec{v}_1 + \vec{e} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} + \vec{e},$$

where $\vec{e}$ is the error in the projection, and again we want to replace $\vec{v}_3$ by some multiple of $\vec{e}$. We have

$$\vec{e} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} = -\vec{w}_3, \quad \text{where} \quad \vec{w}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

so we throw away $\vec{v}_3$ and replace it by $\vec{w}_3$. If we rename $\vec{v}_1$ as $\vec{w}_1$, we now have

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \vec{w}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

where $\vec{w}_1 \perp \vec{w}_2$ and $\vec{w}_1 \perp \vec{w}_3$, but we probably don't have $\vec{w}_2 \perp \vec{w}_3$; in fact, $\vec{w}_2 \cdot \vec{w}_3 = 1 + 0 + 1 + 0 = 2 \neq 0$, so we definitely don't have $\vec{w}_2 \perp \vec{w}_3$. To fix this we need to project one of $\vec{w}_2$ and $\vec{w}_3$ onto the other one and replace the projected vector by the error. Either way of doing this works pretty well. We have $\vec{w}_2 \cdot \vec{w}_2 = 1 + 0 + 1 + 1 = 3$ and $\vec{w}_3 \cdot \vec{w}_3 = 1 + 4 + 1 + 0 = 6$, so the projection of $\vec{w}_3$ onto $\vec{w}_2$ is

$$\frac{\vec{w}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 = \frac{2}{3} \vec{w}_2 = \frac{2}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

and therefore

$$\vec{w}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \vec{e},$$

and we want to replace $\vec{w}_3$ by a multiple of $\vec{e}$. We have

$$3\vec{e} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 1 \\ 2 \end{pmatrix},$$

which is the third vector we need. We now have that

$$\begin{pmatrix} 1 \\ -6 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

are all perpendicular to each other, and we only have to fix the lengths. The dot product of the first vector with itself is 42, and we did the other two earlier, so we finally get that an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ is

$$\frac{1}{\sqrt{42}}\begin{pmatrix}1\\-6\\1\\2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\1\\2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}.$$

If we had projected $\vec{w}_2$ onto $\vec{w}_3$ instead then the answer would have been

$$\frac{1}{\sqrt{21}}\begin{pmatrix}2\\2\\2\\-3\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\1\\2\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}$$

There are many other possible answers. A really cheap one, which no one noticed (and I didn't either until I was almost finished grading them), is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

4. Let A be the matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns. Then

$$A^T A = \begin{pmatrix}1&3&1&1\\1&1&3&1\end{pmatrix}\begin{pmatrix}1&1\\3&1\\1&3\\1&1\end{pmatrix} = \begin{pmatrix}12&8\\8&12\end{pmatrix},$$

and

$$\begin{pmatrix}12&8\\8&12\end{pmatrix}^{-1} = \frac{1}{12\cdot 12-8\cdot 8}\begin{pmatrix}12&-8\\-8&12\end{pmatrix} = \frac{1}{20}\begin{pmatrix}3&-2\\-2&3\end{pmatrix},$$

so

$$P = \frac{1}{20}\begin{pmatrix}1&1\\3&1\\1&3\\1&1\end{pmatrix}\begin{pmatrix}3&-2\\-2&3\end{pmatrix}\begin{pmatrix}1&3&1&1\\1&1&3&1\end{pmatrix} = \frac{1}{20}\begin{pmatrix}1&1\\7&-3\\-3&7\\1&1\end{pmatrix}\begin{pmatrix}1&3&1&1\\1&1&3&1\end{pmatrix},$$

and so we find that the projection matrix P onto the subspace S is

$$P = \frac{1}{20}\begin{pmatrix}2&4&4&2\\4&18&-2&4\\4&-2&18&4\\2&4&4&2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}1&2&2&1\\2&9&-1&2\\2&-1&9&2\\1&2&2&1\end{pmatrix}.$$

We also have that

$$R = 2P - I = \frac{1}{5}\begin{pmatrix}1&2&2&1\\2&9&-1&2\\2&-1&9&2\\1&2&2&1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}5&0&0&0\\0&5&0&0\\0&0&5&0\\0&0&0&5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-4&2&2&1\\2&4&-1&2\\2&-1&4&2\\1&2&2&-4\end{pmatrix}$$

is the reflection matrix through S. The projection of $\vec{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ -1 \end{pmatrix}$ onto S is

$$P\vec{v}_3 = \frac{1}{10}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 9 & -1 & 2 \\ 2 & -1 & 9 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10 \\ 30 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix},$$

and the reflection of $\vec{v}_3$ through S is

$$R\vec{v}_3 = \frac{1}{5}\begin{pmatrix} -4 & 2 & 2 & 1 \\ 2 & 4 & -1 & 2 \\ 2 & -1 & 4 & 2 \\ 1 & 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5 \\ 15 \\ 5 \\ 15 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 3 \end{pmatrix}.$$

The projection should be a linear combination of $\vec{v}_1$ and $\vec{v}_2$, so the fact that it came out to $\vec{v}_1$ exactly, while a little surprising, is consistent. The projection is always the average of the reflection and $\vec{v}_3$ itself, and this could have been used to avoid one of the last two matrix multiplications.

5. If $P = \frac{1}{12}\begin{pmatrix} 5 & 2 & -1 & -1 & 2 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 5 & 5 & 2 & -1 \\ -1 & 2 & 5 & 5 & 2 & -1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 5 & 2 & -1 & -1 & 2 & 5 \end{pmatrix}$ then P is symmetric and

$$P^2 = \frac{1}{144}\begin{pmatrix} 60 & 24 & -12 & -12 & 24 & 60 \\ 24 & 24 & 24 & 24 & 24 & 24 \\ -12 & 24 & 60 & 60 & 24 & -12 \\ -12 & 24 & 60 & 60 & 24 & -12 \\ 24 & 24 & 24 & 24 & 24 & 24 \\ 60 & 24 & -12 & -12 & 24 & 60 \end{pmatrix} = P.$$

Any matrix P for which $P^2 = P = P^T$ is a projection matrix. The trace of P is $\frac{1}{12}(5+2+5+5+2+5) = 2$, so the subspace T that P projects onto is 2-dimensional. Therefore any two rows or columns of P will be a basis for it as long as they are not multiples of each other—in other words, any two of the first three columns or rows should work. But since we also have to find a basis for $T^\perp$, which must be 4-dimensional, let's eliminate. The last three rows obviously drop out, and we have

$$P = \frac{1}{12}\begin{pmatrix} 5 & 2 & -1 & -1 & 2 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 5 & 5 & 2 & -1 \\ -1 & 2 & 5 & 5 & 2 & -1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 5 & 2 & -1 & -1 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 & 2 & -1 & -1 & 2 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 5 & 5 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 6 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

P projects onto its own column space, or its row space since P is symmetric. This gives us a nice basis for T, and a basis for $T^\perp$ comes from the null space of P. We conclude that

$$\textbf{A basis for } T \text{ is} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\textbf{A basis for } T^\perp \text{ is} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

We can get a slightly prettier basis for $T^\perp$ by subtracting one of the first two vectors from the other:

$$\textbf{A basis for } T^\perp \text{ is} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

6. A general way to do this is to take any two independent vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^4$, make them the rows of a 2×4 matrix A, and eliminate to reduced row echelon form to find the null space of A. A basis for this will consist of two independent vectors which we'll call $\vec{w}_1$ and $\vec{w}_2$; these are perdendicular to $\vec{v}_1$ and $\vec{v}_2$. Now let B be the matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$, and let C be the matrix whose rows are (the transposes of) $\vec{w}_1$ and $\vec{w}_2$. Then BC does what we want, by more or less the same logic as in problem 23 in section 3.6. BC is a 4×4 matrix of rank 2, whose column space has basis $\vec{v}_1$ and $\vec{v}_2$ and whose row space has basis $\vec{w}_1$ and $\vec{w}_2$. Its null space must therefore have a basis consisting of two vectors perpendicular to $\vec{w}_1$ and $\vec{w}_2$, and since $\vec{v}_1$ and $\vec{v}_2$ are two such vectors, they are a basis for the null space of BC as well as the column space. Problem 2 contains several examples that work, such as

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 1 & -1 \\ -7 & 4 & 2 & -1 \\ -5 & 8 & 2 & -3 \\ -17 & 11 & 5 & -3 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -4 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ -9 & -8 & -1 & 4 \\ -4 & -17 & 2 & 3 \end{pmatrix}$$