1. A "True or False" question. In this problem, assume $A$ and $B$ are $4 \times 4$ square matrices. Suppose:
   
i) $rref(A) = R$ where $R$ has exactly three leading-1's,
   
ii) $rref(B) = S$ and $S = I_4$. In the left margin, next to each statement, write "T" if the statement is always true and "F" if it's not always true, for any such matrices $A$ and $B$.

   a) $A$ is an invertible matrix.
   
   F

   b) The determinant of $B$ is non-zero.
   
   T

   c) The equation $Ax = b$ has infinitely many solutions $x$ for each $b \in \mathbb{R}^4$.
   
   F

   Since the bottom row of $R$ is all 0's, there are $b$'s for which
   
   $A\hat{x} = b$ is inconsistent and so $\text{no soln}$ for such $b$'s. Otherwise there are $\infty$
   
   soln's!

   d) $\text{Nul}(A) = \text{Nul}(R)$.
   
   T

   e) $\text{rank}(A) = 3$.
   
   T

   f) $\text{Col}(B) = \text{Col}(S)$.
   
   T

   g) The number 0 is an eigenvalue for $A$.
   
   T

   h) The number 0 is an eigenvalue for $B$.
   
   F

   i) $\text{Nul}(B) = \{0\}$.
   
   T

   (the only soln to $B\hat{x} = 0$ is $\hat{x} = 0$; the only vector in $\text{Nul}(B)$ is the zero vector.

   BE CAREFUL: The question does not ask if $\{0\}$ is a BASIS for $\text{Nul}(B)$!

   j) $A$ and $B$ are not row equivalent.
   
   T

   k) $A$ and $B$ are not similar.
   
   T

   Since 0 is an eigenval for one but not the other,
   
   their char. polys are different. But similar matrices do have the same char. polys.

   l) In any basis of $\text{Nul}(A)$, there is exactly one vector.
   
   T
2. Let $B = \begin{bmatrix} 5 & 1 & 2 \\ -5 & 11 & 2 \\ 10 & -4 & 2 \end{bmatrix}$ Here are some facts about $B$:

(i) $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ is in $\text{Nul}(B)$,

(ii) $\lambda = 10$ is an eigenvalue of $B$,

(iii) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $B$.

You should be able to answer these questions without finding any determinants and with maybe just one reference.

a) What is the eigenvalue corresponding to the eigenvector $\mathbf{u}$?

we find $B\mathbf{u} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 8\mathbf{u}$ so $\lambda = 8$

b) What is the characteristic polynomial of $B$ in factored form?

Fact (i) above tells us that $\lambda = 0$ is an eigenvalue of $B$.

Fact (ii) gives that $\lambda = 10$ is a "$\lambda$" $\lambda$ $\lambda$ $\lambda$ $\lambda$ $\lambda$ $\lambda$ $\lambda$ $\lambda$.

Part (a) says $\lambda = 8$ is an eigenvalue of $B$. Since $B$ is $3 \times 3$, its characteristic poly has degree 3, and we have 0, 10, and 8 are roots. In fact they must all be zero and each must be multiplicity 1 (or the degree of the poly would be $> 3$).

So that poly is $(\lambda - 0)(\lambda - 10)(\lambda - 8)$ or $\lambda(\lambda - 10)(\lambda - 8)$

c) Show that $B$ is diagonalizable by exhibiting $P$, $D$ and $P^{-1}$ that have the required properties.

Since each eigenvalue has multiplicity one, their eigenspaces have dimension one also.

We have an eigenvector for each of $\lambda = 0$ and $\lambda = 8$ and we need one

for $\lambda = 10$. Row reduction of $(B - 10I) = \begin{bmatrix} 5 & 1 & 2 \\ -5 & 11 & 2 \\ 10 & -4 & 2 \end{bmatrix}$ yields $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for

null $(B - 10I)$ and hence a basis for the eigenspace $\lambda = 10$

Let $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 1 \end{bmatrix}$; these 3 columns are

eigenvectors corresponding
to $\lambda = 0$, 10, and 8 respectively; hence

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Finally, knowing the eigenvectors corresponding to distinct eigenvalues must form a L.I. set, so we expect $P^{-1}$ to exist and sure enough,

by calculator, $P^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{-1}{8} & \frac{-1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -1 & -2 \\ 1 & -4 & 0 \\ 1 & -4 & 0 \end{bmatrix}$
3. Let \( C = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 7 & -1 \\ 0 & 6 & 2 \end{bmatrix} \).

a) Find the characteristic polynomial of \( C \) in factored form starting from \( \det(C - \lambda I) \); show all your work.

Use the 1st column:
\[
\det \left( \begin{bmatrix} 4 - \lambda & 3 & -1 \\ 0 & 7 - \lambda & -1 \\ 0 & 6 & 2 - \lambda \end{bmatrix} \right)
\]
\[
= (4-\lambda) \begin{vmatrix} 7-\lambda & -1 \\ 6 & 2-\lambda \end{vmatrix}
= (4-\lambda)(7-\lambda)(2-\lambda) + 6
= (4-\lambda)(7-\lambda)(2-\lambda) + 6
= (4-\lambda)(\lambda^2 - 9\lambda + 20)
= (4-\lambda)(\lambda - 4)(\lambda - 5)
\]

b) One of the eigenvalues should have multiplicity 2. Find a basis for the eigenspace of that eigenvalue.

\( \lambda = 4 \) has multiplicity 2.

Now, \( (C - 4I) = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 3 & -1 \\ 0 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

And the nullspace of this matrix is "\( x_1 \) is free, \( x_2 = \frac{1}{2} x_3 \), i.e., all L.C.'s \( x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \)

And a basis for this nullspace is a basis for the eigenspace of \( C \) for \( \lambda = 4 \)

is \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \)

c) Without actually finding \( P, D \) and \( P^{-1} \), explain why \( C \) is diagonalizable. \( D \) exists because we have the eigenvalues 4, 4, and 5; more importantly, the dimension of each eigenspace matches the multiplicity of the eigenvalue and the sum of these is 3, so we can construct \( P \) with corresponding LI columns; \( P^{-1} \) also exists now...
4. Let $M$ be the $4 \times 4$ matrix here. FACT: The product of

$$
\begin{bmatrix}
6 & 1 & 2 & 7 \\
2 & 6 & 2 & -4 \\
2 & 3 & 6 & -3 \\
2 & -4 & 2 & 6
\end{bmatrix}
\text{ and }
\begin{bmatrix}
12 & 7 & 5 & -3 \\
-1 & 2 & 6 & 2 \\
0 & 6 & 7 & 1 \\
7 & 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
10 & 0 & 10 & -2 \\
0 & 0 & 12 & -1 \\
-1 & 0 & 14 & 6 \\
0 & 0 & 0 & 3
\end{bmatrix}

$$
is
\begin{bmatrix}
120 & 56 & 50 & 0 \\
-10 & 38 & 60 & 0 \\
0 & 56 & 70 & -4 \\
70 & 18 & 0 & 0
\end{bmatrix}

(a) Use this fact to find the eigenvalues of $M$, and bases for their respective eigenspaces. Not the fact contains useful and not useful information! 

- The first column of the product says $\lambda = 10$ is an eigenvalue & $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 10$.
- Second column says merely that $\begin{bmatrix} 2 \\ 6 \\ 0 \\ 0 \end{bmatrix}$ is NOT an eigenvector.
- Third column says $\lambda = 10$ is an eigenvalue & $\begin{bmatrix} 5 \\ 6 \\ 2 \\ 0 \end{bmatrix}$ is an eigenvector (so at this point we know dim(eigenspace) is at least 2, because the 2 vectors are a L.I. set).
- Fourth column says $\begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{2}{2} \\ 0 \end{bmatrix}$ is in $\text{Null}(M)$ so $\lambda = 0$ is an eigenvalue, and $\begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{2}{2} \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 0$.
- Fifth column says $\lambda = 4$ is an eigenvalue & $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a corresponding eigenvector.
- Sixth column says $M^6 = 0$ which is not useful here.
- Seventh column says $\lambda = 10$ is an eigenvalue and $\begin{bmatrix} 10 \\ 12 \\ 14 \\ 16 \end{bmatrix}$ is a corresponding eigenvector. 

BUT the vector is a L.I. of the previous two eigenvectors for $\lambda = 10$, and so it tells us NOTHING NEW and does NOT imply the dim. of the eigenspace nor the multiplicity of $\lambda = 10$ is (nor) 3!!

- Eighth column says $\begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{2}{2} \\ 0 \end{bmatrix}$ is NOT an eigenvector (BE CAREFUL).

So we have $\lambda = 10$, $\lambda = 0$, and $\lambda = 4$. Since $M$ is $4 \times 4$, its char poly is degree 4. So at most one of these eigen values has multiplicity > 1. On the other hand, since we've already found a set of 2 L.I. eigenvectors for $\lambda = 10$, its eigenspace must have dim > 2, and since "dim(eigenspace) \leq \text{multiplicity}", the mult. of $\lambda = 10$ is at least 2 also. But it can't be more than 2 since then the char. poly has degree more than 4; the other 3 eigenvalues have eig spaces of exactly 1 and indeed, we even have enough eig vectors to make the bases! SO: here's a table:

<table>
<thead>
<tr>
<th>Eigenvalue $\lambda$</th>
<th>10</th>
<th>0</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis for corresp. Eigenspace</td>
<td>$\begin{bmatrix} -1 \ 1 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \ 0 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0 \ 1 \ 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

(b) Is $M$ diagonalizable? (Y/N)  

(Y)
5. Let \( D = \begin{bmatrix} a & 0 & 1 \\ c & 2 & 0 \\ 1 & 0 & e \end{bmatrix} \) and suppose \( \det(D) = -10 \). Find each of the following:

(a) \( \det(D^3) \)
\[
= \det(D \cdot D \cdot D) = \det(D) \cdot \det(D) \cdot \det(D) = (-10) \cdot (-10) \cdot (-10) = -1000
\]

(b) \( \det(3D) \)
Since each row of \( D \) gets multiplied by 3, and the determinant changes by "3 factors of 3" so, \( \det(3D) = 3 \cdot 3 \cdot 3 \cdot (-10) = -270 \)

(c) \( \det(D+D) \neq \det(D) + \det(D) = -20 \)
In fact, \( \det(D+D) = \det(2D) = 2 \cdot 2 \cdot 2 \cdot \det(-10) = 8 \cdot -10 = -80 \)

(d) \( \det(D^{-1}) = \frac{1}{\det(D)} \) if \( \det(D) \neq 0 \);
So \( \frac{1}{-10} \)

(e) \( |(D)| \)
\[
= \left| \begin{array}{ccc} -10 \\ -10 \\ -10 \end{array} \right| = -10 \\
= \text{absolute value of } -10
\]
\[
= 10
\]

(f) \( |D^T| = \det(D) = (-10) \)

(g) \( ae \)
Use the top row:
\[
\det(D) = ae \left| \begin{array}{cc} 2 & 0 \\ 0 & e \end{array} \right| + 1 \left| \begin{array}{cc} 0 & 2 \\ 1 & 0 \end{array} \right| \\
-10 = ae \cdot 2e + 1 \cdot (-2) \\
-8 = 2ae
\]
\[
ae = -4
\]

(h) \( \det \left( \begin{bmatrix} a & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & e \end{bmatrix} \right) \)
This is a diagonal matrix.
So its det. is \( a \cdot 2 \cdot e \)
\[
= 2ae \\
= -8 \text{ (from part (g))}
\]

\( \text{OR use the middle column!} \)
\[
\det(D) = 2 \left| \begin{array}{c} a \\ 0 \\ 1 \end{array} \right| \\
-10 = 2 (ae - 1) \\
-10 = 2ae - 2 \\
-8 = 2ae \therefore ae = -4 \text{ (again)}
\]