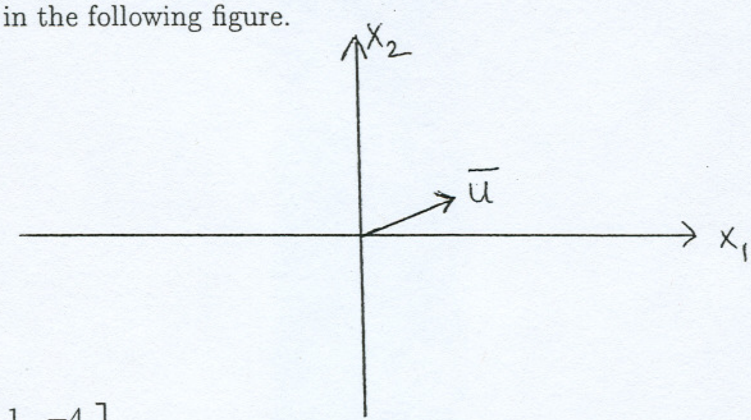
Math 205A Test 2 (50 points)

Name: ______________________________

- Check that you have 7 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (10 points) Short answers: (Show all the calculations needed to get the answers. No explanations needed.)

   (a) Suppose B is a 2×2 matrix and $\vec{u}$ is an eigenvector of B corresponding to the eigenvalue -2. Draw $B\vec{u}$ in the following figure.



   (b) Let $A = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Find all the eigenvalues of A.

(c) Let $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}$. Find a vector of length 7 in the direction of the vector $\vec{u}$.

(d) Compute the orthogonal projection of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ onto the line through $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and the origin.

2. (9 points) Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 3 & -2 & -3 \\ 0 & 0 & 4 & -6 \end{bmatrix}$.

(a) Find a basis for Col A and then state the dimension of Col A.

(b) Is the basis you found in part (a) an orthogonal basis? Explain.

(c) Let $\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 3 \\ -6 \end{bmatrix}$. Find the coordinates of $\vec{x}$ with respect to the basis you found in part (a).

3. (7 points) Let $B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) Is -1 an eigenvalue of B? If so, find a basis for the corresponding eigenspace. If not, explain why not.

(b) Is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ an eigenvector of B? If so, find the corresponding eigenvalue. If not, explain why not.

4. (8 points) Suppose A and B are 5×5 matrices with $\det A = -10$ and $\det(AB^2) = 0$.

(a) Find $\det B$.

(b) Is B invertible? Explain.

(c) What is the largest possible rank of B? What is the smallest possible dimension of Nul B? Provide explanations for your answers.

5. (7 points) Suppose a matrix A can be factored in the form PDP^{-1} where $P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find all the eigenvalues of A and a basis for each eigenspace.

(b) If possible, find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Explain clearly why the basis you find satisfies the two conditions in the definition of a basis. If it is not possible to find such a basis, explain why not.

6. (4 points) Let $H = \left\{ \begin{bmatrix} c \\ b - 2a \\ a + b + c \\ 5b \end{bmatrix} : a, b, c \text{ are real numbers.} \right\}$.

Is H a subspace of $\mathbb{R}^4$? Explain.

7. (5 points) For a certain animal species, there are two life stages: juvenile and adult. For $k \geq 0$, let $\vec{x}_k$ be a vector in $\mathbb{R}^2$ that denotes the population of the species at the end of year k. The first entry in the vector gives the number of juveniles and the second entry gives the number of adults. If A is the 2×2 stage-matrix for the species, the population is given by the equation $\vec{x}_{k+1} = A\vec{x}_k$. Eigenvalues of A are 1.2 and -0.4 and the corresponding eigenvectors are $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ respectively. The initial population is 100 juveniles and 55 adults,i.e., $\vec{x}_0 = \begin{bmatrix} 100 \\ 55 \end{bmatrix}$.

(a) Write the general solution for the population equation, i.e., write an expression for $\vec{x}_k$.

(b) Use you answer in part (a) to discuss the population growth of the species in the long run.