1. (10 points) Short answers: (Show all the calculations needed to get the answers. No explanations needed.)

(a) Suppose $B$ is a $2 \times 2$ matrix and $\vec{u}$ is an eigenvector of $B$ corresponding to the eigenvalue $-2$. Draw $B\vec{u}$ in the following figure.

(b) Let $A = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Find all the eigenvalues of $A$. 

\[ \begin{bmatrix} 2 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \]
(c) Let \( \vec{u} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \end{bmatrix} \). Find a vector of length 7 in the direction of the vector \( \vec{u} \).

(d) Compute the orthogonal projection of \( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \) onto the line through \( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) and the origin.
2. (9 points) Let \( A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 3 & -2 & -3 \\ 0 & 0 & 4 & -6 \end{bmatrix} \).

(a) Find a basis for \( \text{Col } A \) and then state the dimension of \( \text{Col } A \).

(b) Is the basis you found in part (a) an orthogonal basis? Explain.

(c) Let \( \vec{x} = \begin{bmatrix} -3 \\ 0 \\ 3 \\ -6 \end{bmatrix} \). Find the coordinates of \( \vec{x} \) with respect to the basis you found in part (a).
3. (7 points) Let \( B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \).

(a) Is \(-1\) an eigenvalue of \( B \)? If so, find a basis for the corresponding eigenspace. If not, explain why not.

(b) Is \( \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \) an eigenvector of \( B \)? If so, find the corresponding eigenvalue. If not, explain why not.
4. (8 points) Suppose $A$ and $B$ are $5 \times 5$ matrices with $\det A = -10$ and $\det (AB^2) = 0$.

(a) Find $\det B$.

(b) Is $B$ invertible? Explain.

(c) What is the largest possible rank of $B$? What is the smallest possible dimension of $\text{Nul } B$? Provide explanations for your answers.
5. (7 points) Suppose a matrix $A$ can be factored in the form $PDP^{-1}$ where 

\[
P = \begin{bmatrix}
-1 & 1 & -1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

and 

\[
D = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]

(a) Find all the eigenvalues of $A$ and a basis for each eigenspace.

(b) If possible, find a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$. Explain clearly why the basis you find satisfies the two conditions in the definition of a basis. If it is not possible to find such a basis, explain why not.
6. (4 points) Let \( H = \left\{ \begin{pmatrix} c \\ b - 2a \\ a + b + c \\ 5b \end{pmatrix} : a, b, c \text{ are real numbers} \right\}. \)

Is \( H \) a subspace of \( \mathbb{R}^4 \)? Explain.
7. (5 points) For a certain animal species, there are two life stages: juvenile and adult. For \( k \geq 0 \), let \( \vec{x}_k \) be a vector in \( \mathbb{R}^2 \) that denotes the population of the species at the end of year \( k \). The first entry in the vector gives the number of juveniles and the second entry gives the number of adults. If \( A \) is the \( 2 \times 2 \) stage-matrix for the species, the population is given by the equation \( \vec{x}_{k+1} = A \vec{x}_k \). Eigenvalues of \( A \) are 1.2 and \(-0.4\) and the corresponding eigenvectors are \( \begin{bmatrix} 8 \\ 6 \end{bmatrix} \) and \( \begin{bmatrix} -4 \\ 1 \end{bmatrix} \) respectively. The initial population is 100 juveniles and 55 adults, i.e., \( \vec{x}_0 = \begin{bmatrix} 100 \\ 55 \end{bmatrix} \).

(a) Write the general solution for the population equation, i.e., write an expression for \( \vec{x}_k \).

(b) Use your answer in part (a) to discuss the population growth of the species in the long run.