Exam 2 Key

(1)
(a) $\frac{\partial r}{\partial v} = uw - 2v$, $\frac{\partial^2 r}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial r}{\partial v}\right) = w$

$r_w = uv - 2w$, $r_{ww} = -2$

(b) $$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial r}{\partial w}\frac{\partial w}{\partial x}$$

$$= (vw - 2u)\,0 + (uw - 2v)(1) + (uv - 2w)(1)$$

$$= (y+z)(x+y) - 2x - 2z + \left[(y+z)(x+z) - 2x - 2y\right]$$

$$\frac{\partial r}{\partial y} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial r}{\partial w}\frac{\partial w}{\partial y}$$

$$= (vw - 2u)(1) + (uw - 2v)(0) + (uv - 2w)(1)$$

$$= (x+z)(x+y) - 2y - 2z + \left[(y+z)(x+z) - 2x - 2y\right]$$

(2) $\text{div}\,\vec{F} = e^y + z\cos y + \frac{xy}{z}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^y & z\sin y & xy\ln z \end{vmatrix}$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ z\sin y & xy\ln z \end{vmatrix}\hat{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xe^y & xy\ln z \end{vmatrix}\hat{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xe^y & z\sin y \end{vmatrix}\hat{k}$$

$$= (x\ln z - \sin y)\hat{i} - (y\ln z)\hat{j} + (0 - xe^y)\hat{k}$$

③ ⓐ $D\vec{h}(1,-1) = D\vec{f}(\vec{g}(1,-1))\, D\vec{g}(1,-1)$

$$D\vec{f}(3,2) = \begin{bmatrix} 2/y & \frac{-2x}{y^2} \\ 6x-y & -x \end{bmatrix}\Bigg|_{(3,2)} = \begin{bmatrix} 1 & -3/2 \\ 16 & -3 \end{bmatrix}$$

$$D\vec{h}(1,-1) = \begin{bmatrix} 1 & -3/2 \\ 16 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 4 & -16 \end{bmatrix}$$

ⓑ $\vec{h}(1,-1) = \vec{f}(\vec{g}(1,-1)) = \vec{f}(3,2) = (3, 26)$

$$\vec{h}(.99, -1.01) \approx \vec{h}(1,-1) + D\vec{h}(1,-1)\left((.99,-1.01)-(1,-1)\right)$$

$$= \begin{bmatrix} 3 \\ 26 \end{bmatrix} + \begin{bmatrix} -5 & -1 \\ 4 & -16 \end{bmatrix}\begin{bmatrix} -.01 \\ -.01 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 26 \end{bmatrix} + \begin{bmatrix} .06 \\ .12 \end{bmatrix} = \begin{bmatrix} 3.06 \\ 26.12 \end{bmatrix}$$

$$= (3.06,\ 26.12)$$

④ $f(x,y,z) = 2x^2 + 4y^2 + z^2$

$$\vec{\nabla} f\Big|_{(2,-3,-1)} = (4x,\ 8y,\ 2z)\Big|_{(2,-3,-1)} = (8, -24, -2)$$

Plane: $(8,-24,-2)\cdot(x-2,\ y+3,\ z+1) = 0$

$$8(x-2) - 24(y+3) - 2(z+1) = 0$$

$$8x - 16 - 24y - 72 - 2z - 2 = 0$$

$$8x - 24y - 2z = 90$$

(5) A function is continuous at a point $\vec{a}$ if $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ exists and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

We begin by finding the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^4+y^2)}{x^4+y^2}$$ let $u = x^4+y^2$

as $(x,y)\to(0,0)$ $u\to 0$.

$$= \lim_{u\to 0} \frac{1-\cos u}{u} = \frac{1-1}{0} = \frac{0}{0}$$ indeterminate form so we can use L'Hopital's rule.

$$= \lim_{u\to 0} \frac{\sin u}{1} = 0$$

Thus $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^4+y^2)}{x^4+y^2} = 0 = f(0,0)$.

Hence the limit exists.

(6) (a) answer is $\vec{\nabla} T\big|_{(3,-1,0)}$

$$= \left(-2xe^{-x^2-y^2} + (-2x+4)e^{-x^2+4x-y^2+4y-8}, -2ye^{-x^2-y^2} + (-2y+4)e^{-x^2+4x-y^2+4y-8}, 2z\right) \Big|_{@(3,-1,0)}$$

$$= (-6e^{-10} + (-2)e^{-10}, 2e^{-10} + (6)e^{-10}, 0)$$

$$= \boxed{(-8e^{-10}, 8e^{-10}, 0)}$$

(6b) no change in temperature $\Rightarrow D_{\vec{u}} T(3,-1,0) = 0.$

find a $\vec{u}$ such that this is true.

solve: $\vec{\nabla} T \Big|_{(3,-1,0)} \cdot \vec{u} = 0$

$$(-8e^{-10}, 8e^{-10}, 0) \cdot (x, y, z) = 0$$

$$-8e^{-10} x + 8e^{-10} y + 0 = 0$$

$$e^{-10}(-8x + 8y) = 0$$

$$-8x + 8y = 0$$

$$8y = 8x$$

$$y = x$$

take $x = 1 \Rightarrow y = 1$

so one ans is $(1, 1, 0)$