Math 205B Test 2 (45 points)

Name: Solutions

- Check that you have 6 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (8 points) Let $A$ and $B$ be $3 \times 3$ matrices, with $\det A = 2$ and $\det B = -5$. Use properties of determinants to compute:

(a) $\det 4B$

$4^3 \det B = 4^3 \cdot (-5) = -320$

(b) $\det C$ where $C$ is obtained from $A$ by interchanging rows 1 and 3

$\det C = -\det A = -2$

(c) $\det A^T B^T$

$$(\det A^T)(\det B^T) = (\det A)(\det B) = 2(-5) = -10.$$

(d) $\det (AB)^{-1}$ if $AB$ is invertible. Otherwise, explain why $AB$ is not invertible.

$\det(AB) = (\det A)(\det B) = 2(-5) = -10$

$\det(AB) \neq 0$. So $AB$ is invertible.

$$\det(AB)^{-1} = \frac{1}{\det(AB)} = \frac{1}{-10}.$$

2. (6 points) Determine if each of the following sets is a subspace of the appropriate vector space. **If so, find a basis and dimension of the subspace.**

(a) Let $W = \left\{ \begin{bmatrix} a-b+2c \\ 2b+2c+d \\ 4b+4c+2d \\ a+b+4c+d \end{bmatrix} : a, b, c, d \text{ are real numbers.} \right\}$. Is $W$ a subspace of $\mathbb{R}^4$? Explain.

$$\begin{bmatrix} a-b+2c \\ 2b+2c+d \\ 4b+4c+2d \\ a+b+4c+d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix} + c\begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

So $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$. Hence $W$ is a subspace of $\mathbb{R}^4$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 1 & 1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1/2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

: Pivots in first two columns.

A linearly independent subset of the spanning set is

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix} \right\}$ which forms a basis for $W$. So $\dim W = 2$.

(b) Let $W = \left\{ \begin{bmatrix} a+b \\ b-2 \\ b+1 \end{bmatrix} : a, b \text{ are real numbers.} \right\}$. Is $W$ a subspace of $\mathbb{R}^3$? Explain.

The zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in $W$.

For the zero vector to be in $W$, $b-2=0$ and $b+1=0$

i.e. $b=2$ and $b=-1$

which is not possible simultaneously.

So $W$ is not a subspace of $\mathbb{R}^3$.

3. (8 points) Let $A = \begin{bmatrix} 32 & 6 & -10 \\ 1 & 33 & -5 \\ 4 & 12 & 10 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. If possible, find a basis $\mathcal{B}$ for $\mathbb{R}^3$ consisting of the eigenvectors of $A$ and write the $\mathcal{B}$-matrix for $T$. If such a basis does not exist, explain why. (The eigenvalues of $A$ are 15 and 30.)

$$A - 15I = \begin{bmatrix} 17 & 6 & -10 \\ 1 & 18 & -5 \\ 4 & 12 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = 1/2\, x_3 \\ x_2 = 1/4\, x_3 \\ x_3 \text{ free.} \end{matrix}$$

So a basis for eigenspace corresponding to 15 is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$. (Choose $x_3 = 4$).

$$A - 30I = \begin{bmatrix} 2 & 6 & -10 \\ 1 & 3 & -5 \\ 4 & 12 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -3x_2 + 5x_3 \\ x_2, x_3 \text{ free.} \end{matrix}$$

So a basis for eigenspace corresponding to 30 is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, we have three linearly independent eigenvectors of $A$.

So $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of the eigenvectors of $A$.

The $\mathcal{B}$-matrix for $T$ is $\begin{bmatrix} 15 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix}$.

4. (a) (5 points) Find the characteristic polynomial of the matrix $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ and then find all the eigenvalues by solving the charateristic equation.

$$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{bmatrix}\right) = (3-\lambda)(5-\lambda)-8$$

$$= 15-8\lambda+\lambda^2-8$$

$$= 7-8\lambda+\lambda^2$$ is the characteristic polynomial.

$$7-8\lambda+\lambda^2=0$$

$$(\lambda-1)(\lambda-7)=0$$

$\lambda=1$, or $\lambda=7$. Eigenvalues are 1 and 7.

(b) (3 points) The vector $\begin{bmatrix} -7 \\ 7 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$. Find the corresponding eigenvalue.

$$\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -7 \\ 7 \end{bmatrix} = \begin{bmatrix} 14 \\ -14 \end{bmatrix} = -2\begin{bmatrix} -7 \\ 7 \end{bmatrix}.$$

The corresponding eigenvalue is $-2$.

(c) (4 points) Find a non-zero vector in Nul $A$ where $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$.

Solve $A\bar{x} = \bar{0}$

$$\begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 3x_3 - 2x_4$

$x_2 = 5x_3 - 4x_4$

$x_3, x_4$ free.

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A non-zero vector in Nul $A$ is $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}$.

In fact, any linear combination of the vectors in the spanning set will work. (as long as it is not $\bar{0}$.)

(d) (3 points) Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{b}_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Then $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for $\mathbb{R}^3$. If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, then find $\vec{x}$.

$$\bar{x} = \begin{bmatrix} 3 & -2 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

5. (4 points) Suppose $A$ is a $8 \times 11$ matrix. What is the largest possible rank of $A$? What is the smallest possible dimension of Nul $A$? Explain.

The maximum number of pivots A can have is 8 as there are only 8 rows.

So largest possible rank $A$ = max number of pivots $= 8$.

If there are 8 pivots in 8 columns, then 3 columns correspond to free variables.

So smallest possible dim Nul $A$ = min. number of free variables $= 3$.

6. (4 points) Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for a subspace $H$ of $\mathbb{R}^5$. Let $\vec{x}$ be a vector in $H$ that is not in the basis. Is the set $\{\vec{v}_1, \vec{v}_2, \vec{x}\}$ linearly independent? Explain.

Since $\bar{x}$ is in $H$,

$\bar{x} = c_1\bar{v}_1 + c_2\bar{v}_2$ for some scalars $c_1$ and $c_2$.

(as $\{\bar{v}_1, \bar{v}_2\}$ is a basis for $H$).

Also, $\bar{x} \neq \bar{v}_1$ or $\bar{x} \neq \bar{v}_2$.

So the equation $\bar{x} - c_1\bar{v}_1 - c_2\bar{v}_2 = \bar{0}$ shows that the set $\{\bar{v}_1, \bar{v}_2, \bar{x}\}$ is a linearly dependent set.