Math 205B Test 2 (45 points)

Name: ___________________________________________________________________

- Check that you have 6 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (8 points) Let $A$ and $B$ be $3 \times 3$ matrices, with $\det A = 2$ and $\det B = -5$. Use properties of determinants to compute:
   
   (a) $\det 4B$

   (b) $\det C$ where $C$ is obtained from $A$ by interchanging rows 1 and 3

   (c) $\det A^T B^T$

   (d) $\det (AB)^{-1}$ if $AB$ is invertible. Otherwise, explain why $AB$ is not invertible.
2. (6 points) Determine if each of the following sets is a subspace of the appropriate vector space. If so, find a basis and dimension of the subspace.

(a) Let \( W = \left\{ \begin{bmatrix} a-b+2c \\ 2b+2c+d \\ 4b+4c+2d \\ a+b+4c+d \end{bmatrix} : a, b, c, d \text{ are real numbers} \right\} \). Is \( W \) a subspace of \( \mathbb{R}^4 \)? Explain.

(b) Let \( W = \left\{ \begin{bmatrix} a+b \\ b-2 \\ b+1 \end{bmatrix} : a, b \text{ are real numbers} \right\} \). Is \( W \) a subspace of \( \mathbb{R}^3 \)? Explain.
3. (8 points) Let \( A = \begin{bmatrix} 32 & 6 & -10 \\ 1 & 33 & -5 \\ 4 & 12 & 10 \end{bmatrix} \) and \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation given by \( T(\vec{x}) = A\vec{x} \). If possible, find a basis \( B \) for \( \mathbb{R}^3 \) consisting of the eigenvectors of \( A \) and write the \( B \)-matrix for \( T \). If such a basis does not exist, explain why. (The eigenvalues of \( A \) are 15 and 30.)
4. (a) (5 points) Find the characteristic polynomial of the matrix \[
\begin{bmatrix}
3 & 2 \\
4 & 5
\end{bmatrix}
\] and then find all the eigenvalues by solving the characteristic equation.

(b) (3 points) The vector \[
\begin{bmatrix}
-7 \\
7
\end{bmatrix}
\] is an eigenvector of the matrix \[
\begin{bmatrix}
2 & 4 \\
1 & -1
\end{bmatrix}
\]. Find the corresponding eigenvalue.
(c) (4 points) Find a non-zero vector in Nul $A$ where $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$.

(d) (3 points) Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{b}_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Then $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for $\mathbb{R}^3$. If $[\vec{x}]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, then find $\vec{x}$. 
5. (4 points) Suppose $A$ is a $8 \times 11$ matrix. What is the largest possible rank of $A$? What is the smallest possible dimension of Nul $A$? Explain.

6. (4 points) Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis for a subspace $H$ of $\mathbb{R}^5$. Let $\vec{x}$ be a vector in $H$ that is not in the basis. Is the set $\{\vec{v}_1, \vec{v}_2, \vec{x}\}$ linearly independent? Explain.