Mid-term Exam #2
MATH 205, Fall 2014

Name: ____

Instructions: Please answer as many of the following questions as possible. Show all of your work and give complete explanations when requested. Write your final answer clearly. No calculators or cell phones are allowed.

This exam has 5 problems and 100 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	20	
2	20	
3	30	
4	20	
5	10	
TOTAL	**100**	

1. (20 points) Consider the following two matrices and their reduced echelon forms:

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 2 & 7 \\ 3 & 4 & 1 & 5 \end{bmatrix}, \qquad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 2 \\ 2 & 0 & 1 & 7 & -1 \\ 1 & 1 & 2 & 7 & 1 \\ 3 & 4 & 1 & 5 & -2 \end{bmatrix}, \qquad \texttt{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) (4 points) Find rank A and rank B.

SOLUTION: rank $A = 4$ and rank $B = 4$

(b) (8 points) Find a basis for Col A and state the dimension of Col A. How does the subspace Col A compare to the subspace Col B?

SOLUTION: A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 7 \\ 5 \end{bmatrix} \right\}$ and the dimension of Col A is 4. The columns of A and B both span all of $\mathbb{R}^4$, so Col A = Col B.

(c) (8 points) Find a basis for Nul B and state the dimension of Nul B. Is there a relationship between the subspaces Nul B and Nul A? *Give a one sentence explanation.*

SOLUTION: A basis for the null space of B is $\left\{ \begin{bmatrix} -5 \\ -2 \\ 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ and the dimension of Nul B is 1. There is no relationship between the null spaces of A and B since Nul A is a subspace of $\mathbb{R}^4$ while Nul B is a subspace of $\mathbb{R}^5$.

2. (20 points) This problem deals with $\mathbb{P}_2$, the vector space consisting of all polynomials with real coefficients of degree less than or equal to 2.

   (a) (10 points) Determine whether each of the following sets is a basis for $\mathbb{P}_2$. Justify your answer in each case.

      i. $\mathcal{A} = \{\, 1 + t \,,\, t + t^2 \,\}$

      ii. $\mathcal{B} = \{\, 1 \,,\, -1 + t \,,\, 1 - 2t + t^2 \,\}$

   SOLUTION:

      i. The set $\mathcal{A}$ is not a basis for $\mathbb{P}_2$. The dimension of the vectors space is 3 and the set $\mathcal{A}$ has only 2 vectors, therefore it cannot span $\mathbb{P}_2$.

      ii. The set $\mathcal{B}$ is a basis for $\mathbb{P}_2$. There are three vectors in $\mathcal{B}$ and $\dim \mathbb{P}_2 = 3$, so by the Basis Theorem it suffices to check if $\mathcal{B}$ is a linearly independent set. The coordinate vectors of the set $\mathcal{B}$ with respect to the standard basis $\mathcal{E} = \{1, t, t^2\}$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Putting these vectors into a matrix we have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix},$$

      it is immediately obvious that the three columns are linearly independent. Therefore $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

   (b) (10 points) Choose a set from part (a) that is a basis for $\mathbb{P}_2$. Find the coordinates of $p(t) = 6 - 5t + 2t^2$ relative to that basis.

   SOLUTION: The only set that is a basis is $\mathcal{B}$, so I find $[6 - 5t + 2t^2]_{\mathcal{B}}$. That is, I need to find c_1, c_2, c_3 such that

$$6 - 5t + 2t^2 = c_1(1) + c_2(-1 + t) + c_3(1 - 2t + t^2).$$

   Mapping each polynomial to its coordinate vector relative to the standard basis $\mathcal{E} = \{1, t, t^2\}$, this is equivalent to solving the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}.$$

   To solve this, augment the matrix and row reduce,

$$\begin{bmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{reduced echelon form} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

   Then $c_1 = 3$, $c_2 = -1$ and $c_3 = 2$, so $[6 - 5t + 2t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

3. (30 points) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) (5 points) Write the characteristic equation of A and find all eigenvalues of A.

SOLUTION: First, $A - \lambda I_2 = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$. Then the characteristic equation of A is $\lambda^2 - 1 = 0$ or $(\lambda - 1)(\lambda + 1) = 0$. Then the eigenvalues of A are $\lambda = 1$ and $\lambda = -1$.

(b) (10 points) Find a basis for each eigenspace.

SOLUTION:

- $\lambda = -1$:

$$A + I_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \texttt{rref}(A + I_2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then a basis for the eigenspace corresponding to $\lambda = 1$ is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

- $\lambda = 1$:

$$A - I_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \texttt{rref}(A - I_2) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Then a basis for the eigenspace corresponding to $\lambda = 1$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(c) (10 points) Is A diagonalizable? If yes, find P, P^{-1}, D such that $A = PDP^{-1}$.

SOLUTION: The eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent and there are two of them, therefore A is diagonalizable. Then

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) (5 points) Find A^3 using part (c).

SOLUTION: $A^3 = PD^3P^{-1}$, and $D^3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $D^3 = D$, so $A^3 = PDP^{-1} = A$ by part (c). Therefore $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Or we can calculate

$$\begin{aligned}
A^3 &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

4. (20 points) Let $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be defined by $T\big(p(t)\big) = p'(t) + p(0)$ for all $p(t)$ in $\mathbb{P}_2$. Here, $p'(t)$ is the first derivative of $p(t)$.

(a) (10 points) Show that T is a linear transformation.

SOLUTION: Let p and q be vectors in $\mathbb{P}_2$. To begin, show that $T\big((p+q)(t)\big) = T\big(p(t)\big) + T\big(q(t)\big)$. We have

$$\begin{aligned} T\big((p+q)(t)\big) &= \big(p+q\big)'(t) + \big(p+q\big)(0) \\ &= p'(t) + q'(t) + p(0) + q(0) \\ &= p'(t) + p(0) + q'(t) + q(0) \\ &= T\big(p(t)\big) + T\big(q(t)\big). \end{aligned}$$

Next let p be a vector in $\mathbb{P}_2$ and let c be a scalar. Then

$$\begin{aligned} T\big((cp)(t)\big) &= \big(cp\big)'(t) + \big(cp\big)(0) \\ &= cp'(t) + cp(0) \\ &= c\big(p'(t) + p(0)\big) \\ &= cT\big(p(t)\big). \end{aligned}$$

Then $T\big((cp)(t)\big) = cT\big(p(t)\big)$.

(b) (5 points) Find a non-zero vector in the kernel of T.

SOLUTION: I am looking for a polynomial $p(t)$ of degree less than or equal to 2 satisfying $T\big(p(t)\big) = 0$, or $p'(t) + p(0) = 0$. Let $p(t) = at^2 + bt + c$ be an arbitrary element of $\mathbb{P}_2$. Then $p'(t) = 2at + b$. In order to satisfy $p'(t) + p(0) = 0$, it must be that $2at + b + c = 0$. Then $a = 0$ and $b = -c$. So a polynomial in the kernel of T will be $p(t) = t - 1$, for example.

(c) (5 points) Is T a mapping onto $\mathbb{P}_2$ (yes or no)? *Give a one sentence explanation.*

SOLUTION: No, the mapping T is not onto $\mathbb{P}_2$. By the definition of T, for every $\mathbf{x}$ in $\mathbb{P}_2$, the polynomial $T(\mathbf{x})$ has degree one less than the original polynomial $\mathbf{x}$. Therefore no polynomial of degree 2 is in the range of T. Thus T does not map $\mathbb{P}_2$ onto $\mathbb{P}_2$.

(d) (**Bonus 5 points**) Find a basis for the kernel of T and the dimension of the kernel of T.

SOLUTION: From part (b), an element in the kernel of T has the form $p(t) = 0 + bt - b$ for b any real number. Then a basis for the kernel of T is $\{t - 1\}$ and the dimension of the kernel of T is 1.

5. (10 points)

(a) (5 points) State the Rank Theorem.

SOLUTION: The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

(b) (5 points) Find a 3×3 matrix whose null space is equal to its column space, or explain why no such matrix exists. *Write your answer in complete sentences.*

SOLUTION: A 3×3 matrix cannot have its null space equal to its column space because of the rank theorem. The matrix A has three columns. This is an odd number so it is impossible for $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$ to be equal and sum to 3 since both of those quantities are whole numbers.