Mid-term Exam #2
MATH 205, Fall 2014

Name: ____

Instructions: Please answer as many of the following questions as possible. Show all of your work and give complete explanations when requested. Write your final answer clearly. No calculators or cell phones are allowed.

This exam has 5 problems and 100 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	20	
2	20	
3	30	
4	20	
5	10	
TOTAL	**100**	

1. (20 points) Consider the following two matrices and their reduced echelon forms:

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 2 & 7 \\ 3 & 4 & 1 & 5 \end{bmatrix}, \qquad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 2 \\ 2 & 0 & 1 & 7 & -1 \\ 1 & 1 & 2 & 7 & 1 \\ 3 & 4 & 1 & 5 & -2 \end{bmatrix}, \qquad \texttt{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) (4 points) Find $\operatorname{rank} A$ and $\operatorname{rank} B$.

(b) (8 points) Find a basis for $\operatorname{Col} A$ and state the dimension of $\operatorname{Col} A$. How does the subspace $\operatorname{Col} A$ compare to the subspace $\operatorname{Col} B$?

(c) (8 points) Find a basis for $\operatorname{Nul} B$ and state the dimension of $\operatorname{Nul} B$. Is there a relationship between the subspaces $\operatorname{Nul} B$ and $\operatorname{Nul} A$? *Give a one sentence explanation.*

2. (20 points) This problem deals with $\mathbb{P}_2$, the vector space consisting of all polynomials with real coefficients of degree less than or equal to 2.

   (a) (10 points) Determine whether each of the following sets is a basis for $\mathbb{P}_2$. Justify your answer in each case.

      i. $\mathcal{A} = \{\, 1 + t \,,\, t + t^2 \,\}$

      ii. $\mathcal{B} = \{\, 1 \,,\, -1 + t \,,\, 1 - 2t + t^2 \,\}$

   (b) (10 points) Choose a set from part (a) that is a basis for $\mathbb{P}_2$. Find the coordinates of $p(t) = 6 - 5t + 2t^2$ relative to that basis.

3. (30 points) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) (5 points) Write the characteristic equation of A and find all eigenvalues of A.

(b) (10 points) Find a basis for each eigenspace.

(c) (10 points) Is A diagonalizable? If yes, find P, P^{-1}, D such that $A = PDP^{-1}$.

(d) (5 points) Find A^3 using part (c).

4. (20 points) Let $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be defined by $T\big(p(t)\big) = p'(t) + p(0)$ for all $p(t)$ in $\mathbb{P}_2$. Here, $p'(t)$ is the first derivative of $p(t)$.

   (a) (10 points) Show that T is a linear transformation.

   (b) (5 points) Find a non-zero vector in the kernel of T.

   (c) (5 points) Is T a mapping onto $\mathbb{P}_2$ (yes or no)? *Give a one sentence explanation.*

   (d) (**Bonus 5 points**) Find a basis for the kernel of T and the dimension of the kernel of T.

5. (10 points)

   (a) (5 points) State the Rank Theorem.

   (b) (5 points) Find a 3×3 matrix whose null space is equal to its column space, or explain why no such matrix exists. *Write your answer in complete sentences.*