Math 205A Test 2 (50 points)

Name: Solutions

- Check that you have 6 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (12 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

   (a) Find the characteristic polynomial of the matrix $\begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}$. (Do not solve the characteristic equation to find the eigenvalues; simply find the polynomial.)

$$\det\left(\begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -2-\lambda & 1 \\ 5 & 3-\lambda \end{bmatrix}\right) = (-2-\lambda)(3-\lambda) - 5$$

$$= -6 + 2\lambda - 3\lambda + \lambda^2 - 5$$

$$= \lambda^2 - \lambda - 11$$

   (b) Find the distance between the vector $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$.

$$\|\bar{u} - \bar{v}\| = \left\| \begin{bmatrix} 2 \\ -3 \\ -1 \\ 2 \end{bmatrix} \right\| = \sqrt{4+9+1+4} = \sqrt{18}$$

(c) Find the orthogonal projection of the vector $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$\left(\frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}\right) \bar{u} = \frac{3+1}{1+1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

(d) Let $\vec{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ and let $\vec{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$. Then the set $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is a basis for W. For the vector $\vec{y} == \begin{bmatrix} -18 \\ 0 \\ 1 \end{bmatrix}$ in W, find $[\vec{y}]_{\mathcal{B}}$.

$\bar{u}_1 \cdot \bar{u}_2 = 0$. So $\mathcal{B}$ is an orthogonal basis. Hence, we can compute $[\bar{y}]_{\mathcal{B}}$ as follows:

$$\left(\frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\right) = \frac{-50}{25} = -2 \quad , \quad \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} = \frac{75}{25} = 3 .$$

So $[\bar{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(e) For a 5×8 matrix A, Col $A = \mathbb{R}^5$. What is the dimension of Nul A?

Since Col $A = \mathbb{R}^5$, rank $A = 5$.

$8 = \dim \text{Nul } A + \text{rank } A$.

So $\dim \text{Nul } A = 8 - 5 = 3$.

2. (10 points) Let $A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$. The eigenvalues of A are 5 and -3.

(a) Find a basis for the eigenspace corresponding to each eigenvalue of A.

$\underline{\lambda = 5}$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/3 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -\frac{4}{3}x_2 + \frac{1}{3}x_3$, x_2, x_3 free.

So every solution of the eqn. $(A-5I)\bar{x} = \bar{0}$ can be written as $x_2 \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$.

Hence, $\text{Nul}(A-5I) = \text{Span}\left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ and this spanning set is also linearly independent.

So basis for the eigenspace $= \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

$\underline{\lambda = -3}$: $A + 3I = \begin{bmatrix} -4 & -16 & 4 \\ 6 & 16 & -2 \\ 12 & 16 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_3$, $x_2 = \frac{1}{2}x_3$, x_3 free.

Every soln. of $(A+3I)\bar{x} = \bar{0}$ can be written as $x_3 \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$.

So basis for the eigenspace $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(b) Is A diagonalizable? If so, find the matrices P and D so that $A = PDP^{-1}$. If not, explain why not.

Let $P = \begin{bmatrix} -4 & 1 & -2 \\ 3 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Then $A = PDP^{-1}$ and so A is diagonalizable.

3. (6 points) Let A and B be 3×3 matrices, with $\det A = 2$ and $\det B = -5$. Use properties of determinants to compute:

(a) $\det 10A$

$\det 10A = 10^3 \cdot \det A = 2000$

(b) $\det B^T A$

$\det B^T A = (\det B^T)(\det A) = (\det B)(\det A) = -10.$

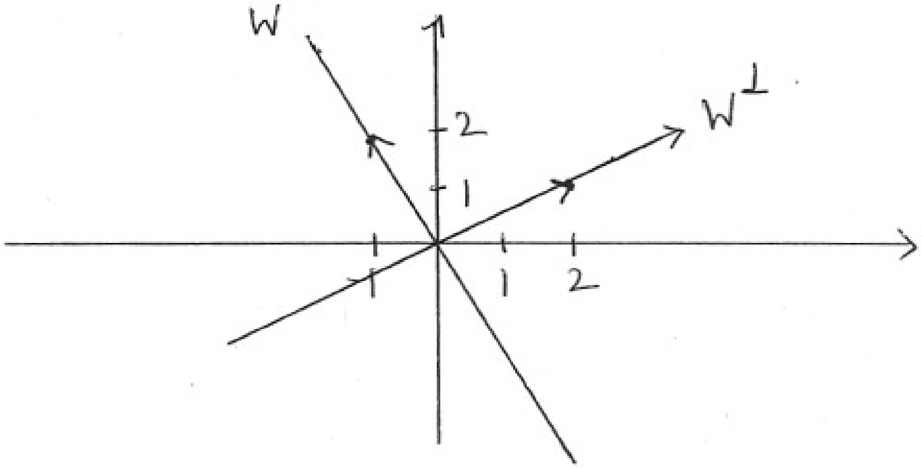
(c) $\det (BA)^{-1}$ if BA is invertible. Otherwise, explain why BA is not invertible.

$\det(BA) = (\det B)(\det A) = -10.$

So $\det(BA)^{-1} = \dfrac{1}{\det(BA)} = -0.1$

4. (6 points) Let $W = \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$.

(a) Draw W and $W^\perp$.



(b) Find a spanning set for $W^\perp$.

$W^\perp$ is a one-dimensional subspace of $\mathbb{R}^2$, as seen in part (a). So a spanning set for $W^\perp$ contains only one vector. We want a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$ ie $-a + 2b = 0$. $a = 2, b = 1$ is one solution to this eqn.

Hence $W^\perp = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

5. (8 points) Let W be a subspace of $\mathbb{R}^5$ and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be an orthogonal basis for W.

(a) Find dim W.

$\dim W = 2$.

(b) Is the set $\{2\vec{v}_1,\ -3\vec{v}_2\}$ an orthogonal set? Explain.

$(2\bar{v}_1)\cdot(-3\bar{v}_2) = 2.(-3).(\bar{v}_1 \cdot \bar{v}_2)$

$= 2(-3).0$ (since $\{\bar{v}_1, \bar{v}_2\}$ is an orthogonal set)

$= 0$.

Hence the set $\{2\bar{v}_1, -3\bar{v}_2\}$ is an orthogonal set.

(c) Is the set $\{2\vec{v}_1,\ -3\vec{v}_2\}$ a basis for W? Use the two conditions in the definition of a basis to explain your answer.

Since the set $\{2\bar{v}_1, -3\bar{v}_2\}$ is an orthogonal set, it is linearly independent

Let $\bar{w}$ be a vector in W. Since $\{\bar{v}_1, \bar{v}_2\}$ is a basis for W, we can find scalars c_1 and c_2 such that

$\bar{w} = c_1\bar{v}_1 + c_2\bar{v}_2$.

But then, $\bar{w} = \frac{c_1}{2}(2\bar{v}_1) - \frac{c_2}{3}(-3\bar{v}_2)$.

Thus, $\bar{w}$ is a linear combination of the vectors $2\bar{v}_1$ and $-3\bar{v}_2$.

Hence, $W = \text{Span}\{2\bar{v}_1, -3\bar{v}_2\}$.

Thus, the set $\{2\bar{v}_1, -3\bar{v}_2\}$ is a basis for W.

6. (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ where A is a 2×2 matrix. Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$. Suppose $T(\vec{u}) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) For each of the vectors $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$ identify whether it is an eigenvector of the matrix A. If a vector is an eigenvector, find the corresponding eigenvalue. If a vector is not an eigenvector, explain why not.

$A\bar{u} = T(\bar{u}) = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ie $A\bar{u} = -3\bar{u}$.

So $\bar{u}$ is an eigenvector of A, with eigenvalue -3.

$A\bar{v} = T(\bar{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ ie $A\bar{v} = 0 \cdot \bar{v}$

So $\bar{v}$ is an eigenvector of A, with eigenvalue 0.

$A(\bar{u}+\bar{v}) = T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $\bar{u}+\bar{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

So $A(\bar{u}+\bar{v}) \neq \lambda(\bar{u}+\bar{v})$ for any scalar λ.

Hence, $\bar{u}+\bar{v}$ is not an eigenvector of A.

(b) Is $A + 3I$ an invertible matrix? Explain.

Since -3 is an eigenvalue of A,

-3 is a solution of the eqn. $\det(A - \lambda I) = 0$.

Hence, $\det(A + 3I) = 0$.

So $A+3I$ is not an invertible matrix.