Math 205A Test 2 (50 points)

Name: ______________________________

- Check that you have 6 questions on two pages.
- Show all your work to receive full credit for a problem.

1. (12 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

   (a) Find the characteristic polynomial of the matrix $\begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}$. (Do not solve the characteristic equation to find the eigenvalues; simply find the polynomial.)

   (b) Find the distance between the vector $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$.

(c) Find the orthogonal projection of the vector $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(d) Let $\vec{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ and let $\vec{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$. Let $W = \text{Span}\,\{\vec{u}_1, \vec{u}_2\}$. Then the set $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is a basis for W. For the vector $\vec{y} == \begin{bmatrix} -18 \\ 0 \\ 1 \end{bmatrix}$ in W, find $[\vec{y}]_{\mathcal{B}}$.

(e) For a 5×8 matrix A, Col $A = \mathbb{R}^5$. What is the dimension of Nul A?

2. (10 points) Let $A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$. The eigenvalues of A are 5 and -3.

(a) Find a basis for the eigenspace corresponding to each eigenvalue of A.

(b) Is A diagonalizable? If so, find the matrices P and D so that $A = PDP^{-1}$. If not, explain why not.

3. (6 points) Let A and B be 3×3 matrices, with $\det A = 2$ and $\det B = -5$. Use properties of determinants to compute:

(a) $\det 10A$

(b) $\det B^T A$

(c) $\det (BA)^{-1}$ if BA is invertible. Otherwise, explain why BA is not invertible.

4. (6 points) Let $W = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

(a) Draw W and $W^{\perp}$.

(b) Find a spanning set for $W^{\perp}$.

5. (8 points) Let W be a subspace of $\mathbb{R}^5$ and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be an orthogonal basis for W.

(a) Find $\dim W$.

(b) Is the set $\{2\vec{v}_1,\ -3\vec{v}_2\}$ an orthogonal set? Explain.

(c) Is the set $\{2\vec{v}_1,\ -3\vec{v}_2\}$ a basis for W? Use the two conditions in the definition of a basis to explain your answer.

6. (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ where A is a 2×2 matrix. Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$. Suppose $T(\vec{u}) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) For each of the vectors $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$ identify whether it is an eigenvector of the matrix A. If a vector is an eigenvector, find the corresponding eigenvalue. If a vector is not an eigenvector, explain why not.

(b) Is $A + 3I$ an invertible matrix? Explain.