ANSWERS

1. [10 pts.] Suppose I claim that the following list will show that there is a one-to-one correspondence between the natural numbers and the real numbers. Prove that I am wrong by describing Cantor’s diagonalization argument, and writing down the first five digits of a decimal number that Cantor’s argument produces that cannot be on this list.

1 ↔ 0.12345678910112131415...
2 ↔ 0.24681012141618202224...
3 ↔ 0.369121518212427303336...
4 ↔ 0.481262024283236404448...
5 ↔ 0.510152025303540455055...
...

Cantor’s argument says: Check the first digit of the first number. If it’s not 2, then 2 is the first digit of your number. If it is 2, then 4 is the first digit of your number. Check the second digit of the second number and do the same (not-2 becomes 2, 2 becomes 4). Same for nth digit of nth number.

Example of the first five digits of a number that cannot appear on above list: 0.22242...

2. [10 pts.] Show that the set of integers \{ ..., -3, -2, -1, 0, 1, 2, 3, ... \} has the same cardinality as the set of natural numbers. Be sure it is clear how all integers and all natural numbers are included.

... -2 -1 0 1 2 ... 1 2 3 4 5 ...
... 5 3 1 2 4 ... 0 1 -1 2 -2

One possible pairing:

Integers to Naturals: (for n > 0, n -> 2n
n <= 0, n -> -2n + 1)

Natural numbers to Integers: (n even -> n/2
n odd -> -(n/2 – ½) )

3. [10 pts.] Give an example of a subset of the natural numbers that can be removed from the set of natural numbers so that the set of remaining things is...

(a) finite. Remove all natural numbers (leaving the null or empty set, with cardinality zero); or remove all natural numbers greater than 5 (leaving the set \{1,2,3,4,5\} also with finite cardinality (cardinality of 5).

(b) infinite. Remove all evens (leaving all odds, which has same cardinality of original set, as we’ve shown).

4. [10 pts.] Is there a one-to-one correspondence between the set \{%, *, #, +, @, A\} and the set \{1, 2, 3, 4, 5\}? If there is, express it. If there is not, explain why not.

No, for finite sets, cardinality is the same as “number of items in set” and one set has 6 elements and the other has only 5, so no 1-1 correspondence can exist. No matter how you try to match, there will always be one element left over in the set \{%, *, #, +, @, A\}.
5. [5 pts.] Put the following sets in order from smallest to largest cardinality. If two sets have the same cardinality, put them in the same box.

S: grains of sand on the earth  
R: real numbers between 0 and 1  
N: natural numbers 1, 2, 3, …  
E: even numbers 2, 4, 6, …  
P: people on the earth

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>N, E</th>
<th>R</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Smallest</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Largest</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

P & S could be switched and it would be ok.

4. [5 pts.] Fill in the following table for the regular solids.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Number of Vertices</th>
<th>Number of Edges</th>
<th>Number of Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

5. [4 pts] a. How many faces does the dual of an octahedron have? 6 faces

[3 pts.] b. Briefly explain the concept of duality of the regular solids. Indicate all the places in this table that show the concept of duality. Two solids are dual when the number of faces of one correspond to the number of vertices of the other, and they have the same number of edges.

6. [4 pts.] a. Is it possible to draw a connected planar graph with 14 vertices and 12 edges? Explain. NO. V – E + F = 2; 14 – 12 + F = 2, so it would have to have NO faces (regions) which is not possible.

[4 pts.] b. Is it possible to draw a connected graph in the plane with 8 vertices, 10 edges, and 5 faces? Explain. NO. 8 – 10 + 5 = 3, and all connected planar graphs must satisfy V – E + F = 2.
[4 pts.] c. A connected, planar graph with 65 faces and 225 vertices has how many edges? 225 - E + 65 = 2, so E = 288.

7. Using ONLY the solid lines of the connected graph below (that is, do not count C as a vertex for part a),

[4 pts.] a. Compute: (number of vertices) – (number of edges) + (number of faces).

(Remember, the outside region counts as a face as well.) 7 – 8 + 3 = 2

[4 pts.] b. If you added an edge from point A to point B, how would the number you just calculated change? Explain very briefly. It wouldn’t change. Add one edge and one face so V – E + F stays the same.

[4 pts.] c. If you added an edge from point C to point D, how would the number you just calculated change? Explain very briefly. It wouldn’t change. Add one edge and one vertex, so V – E + F stays the same.

8. [4 pts.] If you add an edge to Graph 1 to get Graph 2, which changes?
   a. V – E + F
   b. E and V, but not F
   c. E and F, but not V
   d. None of a, b, or c is the correct answer

C is the correct answer


See p. 361-2 in text.
10. **[10 pts.]** “Two sets have the same cardinality if there is a one-to-one correspondence between the contents of one and the contents of the other.” (p. 155)

    a. Suppose you have two *infinite* sets $A$ and $B$ and you are told that there exists a pairing in which each element from $A$ is associated with exactly one element from $B$ and no two elements of $A$ are paired up with the same element of $B$. In this pairing, however, there are elements of $B$ that are not paired up with elements of $A$. Does this failure to find a one-to-one correspondence imply that the cardinality of $A$ and the cardinality of $B$ are not equal? Carefully explain why or why not. **[HINT: You might want to think of some concrete examples of different pairings of two infinite sets $A$ and $B$.]**

    *No it does not. It only shows that this pairing is not one-to-one. Another pairing might be.*

    *SEE p. 147 in text!*

    b. Suppose we replaced the word “infinite” in part a (above) with “finite.” Would that change your answer? Explain.

    *YES, the answer changes. For finite sets, if one pairing fails, all will. There will always be elements of $B$ left over.*