Math 206 — Second Midterm

November 9, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 7 pages including this cover AND IS DOUBLE SIDED. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.
5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.
6. You may use any previously permitted calculator. However, you must state when you use it.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones and hats.
9. Remember that this is a chance to show what you've learned, and that the questions are just prompts.

Problem	Points	Score
1	18	
2	18	
3	17	
4	15	
5	15	
6	17	
Total	100	

1. [18 points] In this problem, $f(x, y)$ is a function with continuous partial derivatives, and

$$\nabla f(20, -100) = (5, -2),$$

and $g(x, y) = f(xy^2, 2x^2y)$.

a. [6 points] What is $\nabla g(5, -2)$?.

Solution: We know that

$$\nabla g(5, -2) = \nabla f(20, -100) * \begin{pmatrix} 4 & -20 \\ -40 & 50 \end{pmatrix} = (100, -200).$$

b. [6 points] Let $n = (4, 2)$. What is the directional derivative $\dfrac{dg}{dn}(5, -2)$?

Solution: We normalize n and get $(4/\sqrt{20}, 2/\sqrt{20})$. Our directional derivative is this normalized vector dotted with our gradient. We get 0.

c. [6 points] Use the above to approximate $g(4, 0)$.

Solution: There are several ways to do this (build a tangent plane is one). I'll do it with directional derivative. We've moved from $(5, -2)$ to $(4, 0)$ so that is the vector $(-1, 2)$. We take the dot product of this with the gradient and get $-500/\sqrt{20}$. We add this to $g(5, -2)$ to get the answer. As this was not given, this is the best we can do $g(5, -2)$.

2. [18 points] In this problem, let $f(x, y) = 36x^3 + 108xy^2 - 507x - 360y$.

a. [6 points] The function $f(x, y)$ has a local maximum at what point?

Solution: We calculate the gradient of f and set equal to zero to get $(108x^2 + 108y^2 - 507, 216xy - 360) = (0, 0)$ We can solve to get $y = 360/216x$. Then we plug in to see that $108x^4 - 507x^2 + 300$. So we use the quadratic formula to get $x^2 = 4$ or $25/36$. This gives that $x = \pm 2$ or $\pm 5/6$. So our four critical points are $(2, 5/6)$, $(-2, -5/6)$, $(5/6, 2)$, and $(-5/6, -2)$.
We then need the Hessian of f which is

$$\begin{pmatrix} 216x & 216y \\ 216y & 216x \end{pmatrix}$$

We then look for the point that gives a negative determinant in the top left, and a positive over all. This means that x is negative, and $y^2 > x^2$. This is the point $(-5/6, -2)$.

b. [6 points] The function $f(x, y)$ has a local minimum at what point?

Solution: We need to find the point that has both determinants positive. This means that x is positive and $x^2 > y^2$. This is the point $(2, 5/6)$.

c. [6 points] The function has two saddle points at what points?

Solution: These are the other two points as their determinants satisfy the saddle point criterion.

3. [17 points] The Apollo theatre has seating area that goes up and down, and is shaped in a semi-circle. The height of the floor is modeled by $f(x,y) = 5x^2 + 2y^2 - 10x$ feet above sea level, and the semi-circular shape is modeled by

$$x^2 + y^2 \leq 36, \quad \text{and} \quad x \geq 0.$$

a. [7 points] What are the critical points of the height of the theatre?

Solution: We need to do the internal critical points where the gradient is zero. This is $(10x - 10, 4y) = (0, 0)$ so it's the point $(1, 0)$. Then we need the boundary with Lagrange multipliers. The corners of the boundary are $(0, -6)$ and $(0, 6)$. The y-axis has gradient $(1, 0)$, so we have the equation $10x - 10 = \lambda$ and $x = 0$, so it's just the corners for that line. The hemicircle is $(10x - 10, 4y) = (2x\lambda, 2y\lambda)$. We solve that $\lambda = 2$ if $y \neq 0$ and that $x = 5$ and $y = \pm\sqrt{11}$. If $y = 0$ then we have that $x = 6$.

b. [5 points] What is the maximum height of the theatre?

Solution: We test all our critical points, and get the heights -5 and 72 and 86 and 120. So the max is at $(6, 0)$.

c. [5 points] What is the minimum height of the theatre?

Solution: This is at the point $(1, 0)$.

4. [15 points] Questions about div, grad, and curl.

a. [5 points] $F : \mathbb{R}^m \to \mathbb{R}^n$ is a function. If we can calculate the gradient of this function, what must be true about m and n?

Solution: To take the gradient, we need that the output is just one number, but it can have any number of inputs. So that m is any positive integer, and $n = 1$.

b. [5 points] $G : \mathbb{R}^m \to \mathbb{R}^n$ is a function. If we can calculate the curl of this function, what must be true about m and n?

Solution: We will need that $m = n = 3$.

c. [5 points] $H : \mathbb{R}^m \to \mathbb{R}^n$ is a function. If we can calculate the the divergence of this function, what must be true about m and n?

Solution: To calculate divergence, we need there to be the same number of outputs as inputs. This just means that $m = n$ as any positive integer.

5. [15 points] You like to buy chips and salsa at the store. Your utility function (your happiness) is $f(x,y) = xy^2 - x - y$ if you buy x units of chips and y units of salsa. The price per units of chips is $5 + 5/x$ if you buy x units, and the price per unit of salsa is always 10. You have \$ 20 to spend. How much chips and salsa should you buy to maximize your happiness?

Solution:

$$
\begin{aligned}
5 + 5x + 10y &= 20 \\
(5,10)\lambda &= (y^2 - 1, 2xy - 1) \\
2y^2 - 2 &= 2xy - 1(\lambda not 0) \\
x &= 2y - 1/y \\
\lambda &= 0 y = 1, x = 1/2.
\end{aligned}
$$

We have the solution when $\lambda = 0$, but we plug in for x in the other case to get $10y - 5/y + 10y = 20$. This gives the quadratic equation $20y^2 - 20y - 5 = 0$ which has solutions $y = \frac{1-\sqrt{2}}{2}$ and $\frac{-1-\sqrt{2}}{2}$. The endpoints are also $(3,0)$ and $(0,3/2)$. Testing these critical points in the function gives the maximum is 0.

6. [17 points] Consider the surface defined by

$$e^x y + 2e^y z - e^z = 0.$$

a. [5 points] What is $\dfrac{dz}{dx}$?

Solution: We take the derivative and get $-\frac{y\,e^x}{2e^y - e^z}$.

b. [5 points] What is $\dfrac{dx}{dy}$?

Solution: We get $-\frac{e^x + 2e^y z}{y\,e^x}$

c. [5 points] What is $\dfrac{dy}{dz}$?

Solution: $-\frac{2e^y - e^z}{e^x + 2e^y z}$

d. [2 points] What is $\dfrac{dz}{dx}\dfrac{dx}{dy}\dfrac{dy}{dz}$?

Solution: Multiplying all together, we get -1.