Math 205 A B Fall 2012 NAME (legibly!) Suggested solutions

EXAM 2 November 9, 2012 Circle Your SECTION: A (8 am) B (9:30 am)

DO NOT WRITE HERE!

1	16
2	11
3	13
4	19
5	18
6	6
7	17
TOTAL	100

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!

1. Suppose an economy with three sectors D, W and Z has consumption matrix $C = \begin{bmatrix} 0.01 & 0.02 & 0.05 \\ 0.10 & 0.06 & 0.07 \\ 0.15 & 0.03 & 0.04 \end{bmatrix}$.

Suppose $\mathbf{x}$, the vector which represents the total number of goods *actually produced* by the three sectors D, W and Z in supplying the final demands of an open sector is $\mathbf{x} = \begin{bmatrix} 3000 \\ 1000 \\ 8000 \end{bmatrix}$.

1A) Let $\mathbf{d}$ be the final demand vector of the open sector. What is the matrix equation involving $\mathbf{x}$, $\mathbf{d}$ and C?

$\boxed{\vec{x} = C\vec{x} + \vec{d}}$ alternatively, $(I_3 - C)\vec{x} = \vec{d}$

1B) Explicitly find $\mathbf{d}$ for the vector $\mathbf{x}$ given above.

$$\begin{bmatrix} 3000 \\ 1000 \\ 8000 \end{bmatrix} = C \begin{bmatrix} 3000 \\ 1000 \\ 8000 \end{bmatrix} + \vec{d} \Rightarrow \vec{d} = (I_3 - C)\vec{x}$$

$$= \underbrace{\begin{bmatrix} .99 & -0.02 & -0.05 \\ -0.10 & .94 & -0.07 \\ -0.15 & -0.03 & .96 \end{bmatrix}}_{\text{Use your calculator to multiply these.}} \begin{bmatrix} 3000 \\ 1000 \\ 8000 \end{bmatrix}$$

$$= \begin{array}{cl} \begin{bmatrix} 2550 \\ 80 \\ 7200 \end{bmatrix} & \begin{array}{l} \leftarrow D \\ \leftarrow W \\ \leftarrow Z \end{array} \end{array}$$

(note! this question involves NO rrefs, NO inverses... unless you mix up the roles of $\vec{x}$ and $\vec{d}$. BE CAREFUL!)

1C) Each unit produced by D requires how many units of Z's product?

Labeling the rows & columns of C $\begin{array}{c|ccc} & D & W & Z \\ D & & & \\ W & & & \\ Z & \boxed{.15} & & \end{array}$, the answer is here So $\underline{0.15}$

1D) Of the total number of units actually produced by sector D, how many of them are consumed by sector Z?

Z consumes 0.05 units of D for each unit of Z produced.

$\begin{array}{c|ccc} & D & W & Z \\ D & & & \boxed{.05} \\ W & & & \\ Z & & & \end{array}$

But 7200 units of Z are produced. (from here)

$\therefore$ $0.05 \times 7200 =$ 360 units of D are consumed by Z

$\dfrac{0.05 \text{ units of D consumed}}{\text{unit of Z produced}} \times 7200 \text{ units of Z produced} = 360 \text{ unit of D consumed.}$

((note well: The # of units 2550 is not involved in Finding this answer, although it's in the question.))

2. Let $A = \begin{bmatrix} 11 & 2 & 0 & -6 \\ 0 & 7 & 0 & 0 \\ 33 & 0 & -4 & -33 \\ 4 & 2 & 0 & 1 \end{bmatrix}$.

2A) Find the characteristic polynomial of A. Show all your work. Be smart and take advantage of the many zeros in this matrix.

char poly of A = $\det\left(\begin{bmatrix} 11-\lambda & 2 & 0 & -6 \\ 0 & 7-\lambda & 0 & 0 \\ 33 & 0 & -4-\lambda & -33 \\ 4 & 2 & 0 & 1-\lambda \end{bmatrix}\right)$. There are several "paths" to finding this det. For example:

$$= +(7-\lambda)\begin{vmatrix} 11-\lambda & 0 & -6 \\ 33 & -4-\lambda & -33 \\ 4 & 0 & 1-\lambda \end{vmatrix}$$

(using the second row expansion, since 3 of the entries are 0's)

$$= (7-\lambda)(-4-\lambda)\begin{vmatrix} 11-\lambda & -6 \\ 4 & 1-\lambda \end{vmatrix}$$

(using the middle column of the previous matrix)

$$= (7-\lambda)(-4-\lambda)\left((11-\lambda)(1-\lambda) - -24\right)$$
$$= (7-\lambda)(-4-\lambda)(11 - 12\lambda + \lambda^2 + 24)$$
$$= (7-\lambda)(-4-\lambda)(\lambda^2 - 12\lambda + 35) = (7-\lambda)(-4-\lambda)(\lambda-7)(\lambda-5)$$
$$\left(= (\lambda-7)^2(\lambda+4)(\lambda-5)\right)$$

2B) Find the eigenvalues of A, and their multiplicities.

Setting the poly. in 2A to 0, we find they are 7 (with multiplicity 2), -4, and 5.

3. Suppose the determinant of some 4x4 matrix M is 3. Next to each of the following matrices, write its determinant.

M^5 $\lvert M^5\rvert = \lvert M\rvert\lvert M\rvert\cdots\lvert M\rvert = \lvert M\rvert^5 = 3^5 = 243$	$5M$ ← each of the 4 rows of M are multiplied by 5. so $= 5\cdot5\cdot5\cdot5\cdot\lvert M\rvert = 5^4\cdot 3 = 1875$	$M + 2M + 7M \neq 3+6+21$ or any such sum! BUT since $M+2M+7M = 10M$, the det is $10^4\cdot 3 = 30{,}000$	$-M$ ← each of the 4 rows of M is multiplied by -1 ∴ $\lvert -M\rvert = (-1)^4\lvert M\rvert = +3$ (NOT -3)
M^{-1} $\lvert M^{-1}\rvert = \frac{1}{\lvert M\rvert} = \frac{1}{3}$	M^T $\lvert M^T\rvert = \lvert M\rvert = 3$	$(M^{-1})^T$ $\lvert (M^{-1})^T\rvert = \lvert (M^{-1})\rvert = \frac{1}{3}$	$((5M)^{-1})^T$ Same as $\det(5M)^{-1}$. Same as $\frac{1}{\det(5M)}$ equals $\frac{1}{1875}$

4. Let $M = \begin{bmatrix} 9 & -8 & 20 \\ 8 & -11 & 40 \\ 2 & -4 & 15 \end{bmatrix}$.

4A) It's a fact that $\lambda = 5$ is an eigenvalue of M. Find a basis for the corresponding eigenspace.

The basis for this eigenspace is the same as a basis of the null space of $M - 5I_3$

so let's find that:

$$M - 5I_3 = \begin{bmatrix} 4 & -8 & 20 \\ 8 & -16 & 40 \\ 2 & -4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since row-equivalent matrices have the same null spaces, and a basis for $\text{null}\left(\begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$, this is also a basis of the eigenspace corresponding to the eigenvalue $\lambda = 5$ of M.

4B) Let $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. Find $M\mathbf{v}$. Then determine if the result says that $\mathbf{v}$ is an eigenvector of M. If so, what's the corresponding eigenvalue?

We find $M\vec{v} = \begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix}$ which is $3\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ is an eigenvector, and $\lambda = 3$ is the corresponding eigenvalue.

4C) The matrix M is diagonalizable. Find P and D for which $M = PDP^{-1}$ where D is a diagonal matrix consisting of the eigenvalues of M, P is invertible, and the column vectors in P are eigenvectors corresponding to the eigenvalues in D.

$P = \begin{bmatrix} 2 & -5 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ these answers are <u>NOT</u> unique of course!

4D) Use your calculator to find P^{-1}. What is it?

for <u>THIS</u> P, it's $\begin{bmatrix} 4 & -7 & 20 \\ 1 & -2 & 6 \\ -1 & 2 & -5 \end{bmatrix}$

5. Suppose that $A = \begin{bmatrix} p & 3 & 5 \\ -7 & x & 0 \\ 4 & 0 & 1 \end{bmatrix}$.

For each matrix below, determine if that matrix is row equivalent to A. If so, find elementary matrices that represent the row operations done to A that turn it into the given matrix. List your matrices in the order you need to multiply them by in order for their product to turn A into the given matrix.

5a) $A_1 = \begin{bmatrix} 3p & 9 & 15 \\ 1 & x & 2 \\ 4 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A = A_1$$

(in this problem, either order is OK)

5b) $A_2 = \begin{bmatrix} -7 & x & 0 \\ 5p & 15 & 25 \\ 1 & 0 & 1/4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = A_2$$

this order means multiply r1 by 5, then swap r1 ↔ r2, then mult. r3 by 1/4

alternatives exist: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = A_2$ ← here we 1st swap r1 & r2. so next, the (new) middle row needs to be multiplied by 5!

NOTE the factor $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$ can actually go anywhere... WHY?

5c) Now, suppose that det(A)=24. What are the determinants of A_1 and A_2?

from 5a we have $3 \cdot 1 \cdot 24 = |A_1|$ ∴

det(A_1) = $\boxed{72}$

from 5b we have

$\frac{1}{4} \cdot -1 \cdot 5 \cdot 24 = |A_2|$

$\therefore |A_2| = -30$

det(A_2) = $\boxed{-30}$

6. Let H be a subspace of $\mathbb{R}^k$ for some k.

6a) What does it mean to say (ie, give the definition) that a set $S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_p\}$ of vectors in H is a *basis* of H?

there are 2 properties S must have

1. Span (S) = H
2. S is a L.I. set.

6b. What is the *dimension* of H? (again, give the *definition*).

that's the # of vectors in any basis of H. (In this case, that's p, if S is a basis of H)

7. Let $K = \begin{bmatrix} 2 & 3 & -6 & -7 \\ 1 & 2 & -1 & -1 \\ 1 & 4 & 7 & 9 \\ 3 & 8 & 5 & 7 \end{bmatrix}$ and let $\mathbf{s} = \begin{bmatrix} 11 \\ 5 \\ 3 \\ 13 \end{bmatrix}$. Label the columns of K as $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$.

You do **not** need to check these three facts:

(1) The vector $\mathbf{s}$ is in Col(K).
(2) The set $\mathcal{B} = \{\mathbf{k}_1, \mathbf{k}_2\}$ is a basis for Col(K).
(3) The set $\mathcal{D} = \{\mathbf{k}_3, \mathbf{k}_4\}$ is a basis for Col(K).

7a) Find $[\mathbf{s}]_{\mathcal{B}}$. (Show any relevant work in all parts of this problem).

This notation means, "find the weights required to write $\vec{s}$ as unique L.C. of the basis vectors in $\mathcal{B}$", that is, solve $\alpha\vec{K}_1 + \beta\vec{K}_2 = \vec{s}$. Then we write $[\vec{s}]_{\mathcal{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

The easiest way to do this is to find

$$\text{rref}\left(\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 5 \\ 1 & 4 & 3 \\ 3 & 8 & 13 \end{array}\right]\right) = \text{rref}\left(\left[\vec{K}_1\ \vec{K}_2 \,\middle|\, \vec{s}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} \alpha = 7 \\ \beta = -1 \end{cases} \therefore [\vec{s}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

in 7a: NOTE WELL : rrefing this:

$$\left[\begin{array}{ccc|c} 2 & 3 & 11 & 0 \\ 1 & 2 & 5 & 0 \\ 1 & 4 & 3 & 0 \\ 3 & 8 & 13 & 0 \end{array}\right]$$ is for solving

"$\alpha\vec{K}_1 + \beta\vec{K}_2 + z\vec{s} = \vec{0}$", which is <u>NOT</u> what this problem is asking for. HERE, the rref tells us

$$\begin{bmatrix} \alpha \\ \beta \\ z \end{bmatrix} = z\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$ where z is free.

$\left\{\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for nul$([\vec{K}_1\ \vec{K}_2\ \vec{s}])$. But THIS IS <u>NOT</u> WHAT you are looking for here, and

$\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$ is <u>NOT</u> $[\vec{s}]_{\mathcal{B}}$!

(although it's related, yes)

7b) Find $[\mathbf{s}]_{\mathcal{D}}$.

now solve $l\vec{K}_3 + m\vec{K}_4 = \vec{s}$, represented as

$$\left[\begin{array}{cc|c} -6 & -7 & 11 \\ -1 & -1 & 5 \\ 7 & 9 & 3 \\ 5 & 7 & 13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -24 \\ 0 & 1 & 19 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{s} = -24\vec{K}_3 + 19\vec{K}_4 \Rightarrow [\vec{s}]_{\mathcal{D}} = \begin{bmatrix} -24 \\ 19 \end{bmatrix}$$

7c) Find $[\mathbf{k}_1]_{\mathcal{B}}$.

Since $1\vec{K}_1 + 0\vec{K}_2 = \vec{K}_1$ we immediately have

$$[\vec{K}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

7d) Find the dimension of Col(K).

since bases for Col(K) have 2 vectors in them, the answer is 2

7e) Find the dimension of Nul(K).

since $4 = \dim(\text{Col}(K)) + \dim(\text{Nul}(K))$

we have $\dim(\text{Nul}(K)) = 2$ also.