MATH 205A
EXAM II
Nov 9, 2007

NAME

$$[A|I] = \left[\begin{array}{ccccc|cccc} 3 & -4 & 3 & 9 & -37 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -4 & 12 & 0 & 1 & 0 & 0 \\ -2 & 4 & 2 & -11 & 40 & 0 & 0 & 1 & 0 \\ \underbrace{4 \quad 0 \quad 20 \quad -7 \quad 10}_{A} & & & & & 0 & 0 & 0 & 1 \end{array}\right]$$

ROW EQUIV:

(RREF:)

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 5 & 0 & -1 & 0 & -14/3 & 7/6 & 5/6 \\ 0 & 1 & 3 & 0 & 4 & 0 & -29/3 & 8/3 & 4/3 \\ 0 & 0 & 0 & 1 & -2 & 0 & -8/3 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{array}\right]$$

$$[B|I] = \left[\begin{array}{ccccc|cccc} -37 & -4 & 3 & 3 & 9 & 1 & 0 & 0 & 0 \\ 12 & 1 & 0 & 3 & -4 & 0 & 1 & 0 & 0 \\ 40 & 4 & -2 & 2 & -11 & 0 & 0 & 1 & 0 \\ \underbrace{10 \quad 0 \quad 4 \quad 20 \quad -7}_{B} & & & & & 0 & 0 & 0 & 1 \end{array}\right]$$

ROW EQUIV:

(RREF:)

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & -1/2 & 0 & 4/3 & -1/3 & -1/6 \\ 0 & 1 & 0 & 3 & 2 & 0 & -15 & 4 & 2 \\ 0 & 0 & 1 & 5 & -1/2 & 0 & -10/3 & 5/6 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{array}\right]$$

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1) be neat
2) show your work
3) read the questions
4) GOOD LUCK!

1. Let A be as the matrix given on the front cover of this exam; note the RREF form of $[A|I_4]$ is given there.

1A. Let $\mathbf{b}$ be a column vector in $\mathbf{R}^4$ with entries b_1, b_2, b_3, and b_4. What conditions, if any, are there on the b_i's so that $\mathbf{b}$ is in $\mathrm{Col}(A)$?

The RREF of $[A|I]$ shows $\vec{b} \in Col(A) \Leftrightarrow \boxed{0 = b_1 - \tfrac{2}{3}b_2 + \tfrac{7}{6}b_3 - \tfrac{1}{6}b_4}$

1B. Find a basis for $\mathrm{Col}(A)$ using the ideas presented in class.

The pivot cols in RREF A are cols 1, 2, and 4, so the corresponding columns of A form a basis of $Col(A)$. The basis is $\left\{ \begin{bmatrix} 3 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -4 \\ -11 \\ -7 \end{bmatrix} \right\}$.

1C. Express the fifth column vector in A as a linear combination of the basis vectors in (1B), and verify that the linear combination is correct by evaluating it; show the work.

The RREF shows easily that col 5 $= -1(\text{col } 1) + 4(\text{col } 2) - 2\,\text{col}(4)$;

these same weights show $\begin{bmatrix} -37 \\ 12 \\ 40 \\ 10 \end{bmatrix} = -1\begin{bmatrix} 3 \\ 0 \\ -2 \\ 4 \end{bmatrix} + 4\begin{bmatrix} -4 \\ 1 \\ 4 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 9 \\ -4 \\ -11 \\ -7 \end{bmatrix}$

Let's CHECK: $= \begin{bmatrix} -3 - 16 - 18 \\ 0 + 4 + 8 \\ 2 + 16 + 22 \\ -4 + 0 + 14 \end{bmatrix} = \begin{bmatrix} -37 \\ 12 \\ 40 \\ 10 \end{bmatrix}$ AS DESIRED!

1D. Find a basis for $\mathrm{Nul}(A)$ using the technique presented in class.

The RREF shows soln's of $A\vec{x} = \vec{0}$ are

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5x_3 + x_5 \\ -3x_3 - 4x_5 \\ x_3 + 0x_5 \\ 0x_3 + 2x_5 \\ 0x_3 + 1x_5 \end{bmatrix} = x_3\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ where x_3 and x_5 are FREE $\therefore \left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ form a basis

1E. Let B be as the matrix given on the front cover of the exam. Note that B has the same columns as A, just in a different order. In terms of linear combinations, give a good argument why $\mathrm{Col}(B)$ and $\mathrm{Col}(A)$ must be identical. (This is a "general truth").

A vector $\vec{v}$ is in $Col(A) \Leftrightarrow \vec{v}$ is some L.C. of the columns of A. Once such a L.C. is found, the terms can be rearranged (since addition is commutative) to match the order in which A's columns appear in B, showing $\vec{v}$ is a L.C. of the columns of B. Any vector in $Col(A)$ is in $Col(B)$ and vice versa, so $Col(A) = Col(B)$.

1F. Is it a coincidence that the last row of RREF form of $[B|I_4]$ is the same as that for $[A|I_4]$? Explain your answer.

No coincidence: since $Col(A) = Col(B)$, any conditions $b_1, \ldots, b_4$ must satisfy for $\vec{b}$ to be in one columnspace must be the same conditions for $\vec{b}$ to be in the other.

(here $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix}$)

2. Again let B be as the matrix given on the front cover of the exam, and suppose $T : \mathbf{R}^5 \to \mathbf{R}^4$ is defined by $T(\mathbf{x}) = B\mathbf{x}$.

2A. Find a basis for the kernel of T using the method discussed in class.

We know $Ker(T) = Nul(B)$. $\vec{x} \in Nul(B) \Leftrightarrow \vec{x} = x_4 \begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$ where x_4 & x_5 are FREE.

Thus a basis is $\left\{ \begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2B. Find a basis for the image of T (or, *range* as the book would say) using the theory developed in class.

We know $Im(T) = Col(B)$ and a basis for the latter is

$$\left\{ \begin{bmatrix} -37 \\ 12 \\ 40 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 4 \end{bmatrix} \right\}$$

2C. Is T one-to-one? If so, explain. If not, give a concrete example showing why not.

<u>NO</u> $T\left(\begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}\right)$ (both are $\vec{0}$ since these two vectors are in the Kernel of T)

yet $\begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$ (so we found two vectors $\vec{v}_1$ & $\vec{v}_2$ st $T(\vec{v}_1) = T(\vec{v}_2)$ yet $\vec{v}_1 \neq \vec{v}_2$ there are ∞-many counter examples, of course!)

2D. Is T onto $\mathbf{R}^4$? If so, explain. If not, find a vector not in the image and explain how you found it.

No. There are vectors $\vec{b}$ in $\mathbb{R}^4$ for which $T(\vec{x}) = \vec{b}$ has no soln. Any vector not in $Col(B)$ will do; we need to find $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix}$ for which $B\vec{x} = \vec{b}$ is inconsistent, and again this happens if $b_1 - \frac{2}{3}b_2 + \frac{7}{6}b_3 - \frac{1}{6}b_4 \neq 0$.
An easy example is $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$!

2E. Are Nul(B) and Nul(A) equal? Hint: can you express the basis vectors for Nul(A) as linear combinations of the basis vectors for Nul(B)?

It's obvious that $\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (a basis vector of $Nul(A)$) can't be expressed as a L.C. of the basis vectors $\begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$ we found in (2A) for $Nul(B)$: if you try to set $\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$ the only choice for α & β which make the last two rows on both sides equal is $\alpha = 0$ & $\beta = 0$, but then this choice makes the L.C on the right equal to $\vec{0}$, not this. So, $\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is <u>NOT</u> in $Nul(B)$, but of course it is in $Nul(A)$. $\therefore$ $Nul(A) \neq Nul(B)$.

3. Let $C = \begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

3A. Find C^{-1} by the "$[C|I] \sim [I|C^{-1}]$" method we developed in class.

$$\left[\begin{array}{cccc|cccc} 0 & 12 & 4 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

R1 replaced by R1 − 12·R2 AND R3 replaced by R3 − 3·R2

changes in BOLD

$$\left[\begin{array}{cccc|cccc} \mathbf{0} & \mathbf{0} & 4 & -1 & \mathbf{1} & \mathbf{-12} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

R1 becomes R1 − 4·R3

$$\left[\begin{array}{cccc|cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{-4} & \mathbf{0} \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap rows as nec. and change the sign

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 4 & 0 \end{array}\right]$$

since this is I_4, this is C^{-1}

(NOTE there are lots of "orders" in which this row reduction can be done.)

3B. What elementary matrix E changes C into $\begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 25 & 8 & -2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ when EC is computed?

Row 2 of C has been replaced with Row 2 + 2 · Row 1. Doing this to I_4 yields $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Let D be the matrix $\begin{bmatrix} 13 & 7 & 0 & 15 \\ 11 & -6 & 5 & 8 \\ 0 & 3 & 0 & 6 \\ 1 & 2 & 0 & 4 \end{bmatrix}$.

4A. Find $\det(D)$, taking advantage of the 0's in the best possible way. Show all your work!

one "easy path": $\det(D) = -5 \begin{vmatrix} 13 & 7 & 15 \\ 0 & 3 & 6 \\ 1 & 2 & 4 \end{vmatrix} = -5\left(13\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 7 & 15 \\ 3 & 6 \end{vmatrix}\right)$

$= -5\left(13 \cdot 0 + 1 \cdot (42 - 45)\right)$

$= -5 \cdot -3 = 15$

4B. Find $\det(DD)$.

$= \det(D) \cdot \det(D) = 15 \cdot 15 = 225$

4C. Find $\det(D^T)$.

$= \det(D) = 15$

4D. Find $\det(D^{-1})$.

$= \dfrac{1}{\det(D)} = \dfrac{1}{15}$

5. Let S be the vector spaces of sequences of real numbers as we discussed in class, so

$$\mathbf{S} = \{\mathbf{s} = (s_1,\ s_2,\ s_3,\ \ldots) \mid \text{each } s_i \text{ is a real number}\}.$$

Define $T : \mathbf{S} \to \mathbf{S}$ by $T(\mathbf{s}) = ({s_1}^2,\ {s_2}^2,\ {s_3}^2,\ \ldots)$.

5A. Find $T((1,\ 3,\ 5,\ 7,\ \ldots))$. (write out the first four members of the new sequence).

$$= (1^2, 3^2, 5^2, 7^2, \cdots) = (1, 9, 25, 49, \cdots)$$

5B. Show that T is not a linear transformation by using a "concrete" counterexample.

we will show $T(\vec{v}_1 + \vec{v}_2) \neq T(\vec{v}_1) + T(\vec{v}_2)$ in general, let $\vec{v}_1 = (1, 3, 5, 7, \cdots)$ and $\vec{v}_2 = (1, 1, 1, \cdots)$

then $T(\vec{v}_1 + \vec{v}_2) = T((1, 3, 5, 7, \cdots) + (1, 1, 1, 1, \cdots)) = T((2, 4, 6, 8, \cdots)) = (4, 16, 36, 64, \cdots)$

while $T(\vec{v}_1) + T(\vec{v}_2) = T((1, 3, 5, 7, \cdots)) + T((1, 1, 1, 1, \cdots)) = (1, 9, 25, 49, \cdots) + (1, 1, 1, 1, \cdots)$

$= (2, 10, 26, 50, \cdots)$; this is NOT $(4, 16, 36, 64, \cdots)$

(N.B. there are lots of concrete counterexamples here, OF COURSE!)

6. Now define $Q : \mathbf{S} \to \mathbf{S}$ by $Q((s_1,\ s_2,\ s_3,\ \ldots)) = (s_3,\ s_4,\ s_5,\ \ldots)$, so Q just "drops" the original first two elements of the sequence $\mathbf{s}$ and shifts all the remaining members of the sequence to the left by two positions. Now Q is indeed a linear transformation; you don't have to prove it. BUT:

6A. Find $Q((1,\ 3,\ 5,\ 7,\ \ldots))$. (write out the first four members of the new sequence).

$$= (5, 7, 9, 11, \cdots)$$

6B. Describe the kernel of Q: All sequences $\mathbf{s}$ of the form $\mathbf{s} = (s_1,\ s_2,\ s_3,\ s_4\ \ldots)$ for which ...

$\ldots\ Q((s_1, s_2, s_3, s_4, \cdots)) =$ the zero vector, $(0, 0, 0, 0, \cdots)$.

Now, because

$Q((s_1, s_2, s_3, s_4, \cdots)) = (s_3, s_4, s_5, s_6, \cdots)$ we

must have $s_3 = s_4 = s_5 = s_6 = \cdots = 0$ in order for $\vec{s}$ to be in $Ker(Q)$,

but of course the values of s_1 & s_2 can be ANY real #'s.

(therefore $Ker(Q) = \{\vec{s} = (s_1, s_2, s_3, \cdots) \in \mathbf{S} \mid \vec{s}$ is of the form $(s_1, s_2, 0, 0, 0, \cdots)$ where s_1 & s_2 are arbitrary real #'s$\}$; Ker Q is a 2-D V.S.)

7. Explain why the subset of vectors $H = \left\{ \begin{bmatrix} \sin(\alpha+\beta) \\ \alpha \\ \beta \end{bmatrix} \in \mathbf{R}^3 \mid \alpha, \beta \in \mathbf{R} \right\}$ cannot be closed under vector addition with a concrete counter example. Hint: let $\alpha = \pi/2$ and $\beta = 0$; add the resulting vector to itself. Why isn't the sum in H?

Let $\alpha = \pi/2$ and $\beta = 0$; then $\vec{v} = \begin{bmatrix} \sin(\alpha+\beta) \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}$ is in H.

now $\vec{v} + \vec{v} = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$, which can't be in H because

there are NO values of x for which $\sin(x) = 2$ since $-1 \le \sin x \le 1$ for all $x \in \mathbb{R}$.

(alternatively: to write $\begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$ as $\begin{bmatrix} \sin(\alpha'+\beta') \\ \alpha' \\ \beta' \end{bmatrix}$ we must set $\alpha' = \pi$ and $\beta' = 0$.

but then the top element would be $\sin \pi = 0$ instead of 2;

there is no pair α', β' for which $\begin{bmatrix} \sin(\alpha'+\beta') \\ \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$)

8. For each of the following vector spaces, give an obvious basis, or explain why there is no basis, and then give the dimension of the vector space:

8A. $\mathbf{R}^3$

(obvious) BASIS

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

DIMENSION

3

8B. $\{0\}$

NO basis: the only set of vectors in $\{\vec{0}\}$ which spans this V.S. is $\{\vec{0}\}$ itself, but this set is not L.I.

0

8C. $\mathbf{P}_3$

$\{1, t, t^2, t^3\}$

or, maybe $\{1, t+1, t^2+t+1, t^3+t^2+t+1\}$

but NOT $\{a+bt+ct^2+dt^3\}$!!

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