MATH 205A
EXAM II
NOV 9, 2007

NAME ☐

$$[A|I] = \left[\begin{array}{c|c} \underbrace{\begin{matrix} 3 & -4 & 3 & 9 & -37 \\ 0 & 1 & 3 & -4 & 12 \\ -2 & 4 & 2 & -11 & 40 \\ 4 & 0 & 20 & -7 & 10 \end{matrix}}_{A} & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{array}\right]$$

ROW EQUIV:

(RREF:)

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 5 & 0 & -1 & 0 & -14/3 & 7/6 & 5/6 \\ 0 & 1 & 3 & 0 & 4 & 0 & -29/3 & 8/3 & 4/3 \\ 0 & 0 & 0 & 1 & -2 & 0 & -8/3 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{array}\right]$$

$$[B|I] = \left[\begin{array}{c|c} \underbrace{\begin{matrix} -37 & -4 & 3 & 3 & 9 \\ 12 & 1 & 0 & 3 & -4 \\ 40 & 4 & -2 & 2 & -11 \\ 10 & 0 & 4 & 20 & -7 \end{matrix}}_{B} & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{array}\right]$$

ROW EQUIV:

(RREF:)

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & -1/2 & 0 & 4/3 & -1/3 & -1/6 \\ 0 & 1 & 0 & 3 & 2 & 0 & -15 & 4 & 2 \\ 0 & 0 & 1 & 5 & -1/2 & 0 & -10/3 & 5/6 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{array}\right]$$

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1) be neat
2) Show your work
3) read the questions
4) GOOD LUCK!

1. Let A be as the matrix given on the front cover of this exam; note the RREF form of $[A|I_4]$ is given there.

1A. Let $\mathbf{b}$ be a column vector in $\mathbf{R}^4$ with entries b_1, b_2, b_3, and b_4. What conditions, if any, are there on the b_i's so that $\mathbf{b}$ is in $\text{Col}(A)$?

1B. Find a basis for $\text{Col}(A)$ using the ideas presented in class.

1C. Express the fifth column vector in A as a linear combination of the basis vectors in (1B), and verify that the linear combination is correct by evaluating it; show the work.

1D. Find a basis for $\text{Nul}(A)$ using the technique presented in class.

1E. Let B be as the matrix given on the front cover of the exam. Note that B has the same columns as A, just in a different order. In terms of linear combinations, give a good argument why $\text{Col}(B)$ and $\text{Col}(A)$ must be identical. (This is a "general truth").

1F. Is it a coincidence that the last row of RREF form of $[B|I_4]$ is the same as that for $[A|I_4]$? Explain your answer.

2. Again let B be as the matrix given on the front cover of the exam, and suppose $T : \mathbf{R}^5 \to \mathbf{R}^4$ is defined by $T(\mathbf{x}) = B\mathbf{x}$.

2A. Find a basis for the kernel of T using the method discussed in class.

2B. Find a basis for the image of T (or, *range* as the book would say) using the theory developed in class.

2C. Is T one-to-one? If so, explain. If not, give a concrete example showing why not.

2D. Is T onto $\mathbf{R}^4$? If so, explain. If not, find a vector not in the image and explain how you found it.

2E. Are $\text{Nul}(B)$ and $\text{Nul}(A)$ equal? Hint: can you express the basis vectors for $\text{Nul}(A)$ as linear combinations of the basis vectors for $\text{Nul}(B)$?

3. Let $C = \begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

3A. Find C^{-1} by the "$[C|I] \sim [I|C^{-1}]$" method we developed in class.

3B. What elementary matrix E changes C into $\begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 25 & 8 & -2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ when EC is computed?

4. Let D be the matrix $\begin{bmatrix} 13 & 7 & 0 & 15 \\ 11 & -6 & 5 & 8 \\ 0 & 3 & 0 & 6 \\ 1 & 2 & 0 & 4 \end{bmatrix}$.

4A. Find $\det(D)$, taking advantage of the 0's in the best possible way. Show all your work!

4B. Find $\det(DD)$.

4C. Find $\det(D^T)$.

4D. Find $\det(D^{-1})$.

5. Let **S** be the vector spaces of sequences of real numbers as we discussed in class, so

$$\mathbf{S} = \{\mathbf{s} = (s_1,\ s_2,\ s_3,\ \dots) \mid \text{each } s_i \text{ is a real number}\}.$$

Define $T : \mathbf{S} \to \mathbf{S}$ by $T(\mathbf{s}) = ({s_1}^2,\ {s_2}^2,\ {s_3}^2,\ \dots)$.

5A. Find $T((1,\ 3,\ 5,\ 7,\ \dots))$. (write out the first four members of the new sequence).

5B. Show that T is not a linear transformation by using a "concrete" counterexample.

6. Now define $Q : \mathbf{S} \to \mathbf{S}$ by $Q((s_1,\ s_2,\ s_3,\ \dots)) = (s_3,\ s_4,\ s_5,\ \dots)$, so Q just "drops" the original first two elements of the sequence **s** and shifts all the remaining members of the sequence to the left by two positions. Now Q is indeed a linear transformation; you don't have to prove it. BUT:

6A. Find $Q((1,\ 3,\ 5,\ 7,\ \dots))$. (write out the first four members of the new sequence).

6B. Describe the kernel of Q: All sequences **s** of the form $\mathbf{s} = (s_1,\ s_2,\ s_3,\ s_4\ \dots)$ for which ...

7. Explain why the subset of vectors $H = \left\{ \begin{bmatrix} \sin(\alpha+\beta) \\ \alpha \\ \beta \end{bmatrix} \in \mathbf{R}^3 \mid \alpha, \beta \in \mathbf{R} \right\}$ cannot be closed under vector addition with a concrete counter example. Hint: let $\alpha = \pi/2$ and $\beta = 0$; add the resulting vector to itself. Why isn't the sum in H?

8. For each of the following vector spaces, give an obvious basis, or explain why there is no basis, and then give the dimension of the vector space:

8A. $\mathbf{R}^3$

8B. $\{\mathbf{0}\}$

8C. $\mathbf{P}_3$