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**Instructions**
- Circle your answers.
- Be neat.
- Read all questions.
- Show all work.

**Advice**
- Good luck!
1. Consider the system of equations:

\[
\begin{align*}
  x_1 + 4x_2 - 10x_3 + x_4 + 5x_5 &= b_1 \\
  5x_1 + 21x_2 - 53x_3 + 10x_4 + 32x_5 &= b_2 \\
  2x_1 + 11x_2 - 29x_3 + 18x_4 + 32x_5 &= b_3 \\
  3x_1 + 14x_2 - 36x_3 + 11x_4 + 27x_5 &= b_4
\end{align*}
\]

Let \( A \) be the matrix of coefficients for this system and let \( c_i \) denote the \( i^{th} \) column of \( A \). It's true that \([A|I_4]\) is row equivalent to:

\[
\begin{bmatrix}
  1 & 0 & 2 & 0 & -4 & 268 & -61 & 19 & 0 \\
  0 & 1 & -3 & 0 & 2 & -70 & 16 & -5 & 0 \\
  0 & 0 & 0 & 1 & 1 & 13 & -3 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 33 & -8 & 2 & 1
\end{bmatrix}
\]

a) Which columns of \( A \) form a basis for \( \text{Col}(A) \)? (Answer using \( c_1, c_2, \text{etc} \), instead of writing out the columns).

\( \hat{c}_1, \hat{c}_2 \text{ and } \hat{c}_4 \) (this is NOT the only answer!!)

b) Express each column \( c_i \) of \( A \) which is not in your basis as a linear combination (LC) of the basis vectors. (Again, answer using \( c_1, c_2, \text{etc} \), instead of writing out the columns).

\[
\hat{c}_3 = 2\hat{c}_1 - 3\hat{c}_2 \\
\text{and} \\
\hat{c}_5 = -4\hat{c}_1 + 2\hat{c}_2 + 1\hat{c}_4
\]

c) Use part (b) to show how \( b = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 + \alpha_4 c_4 + \alpha_5 c_5 \), a LC of all five columns, can be expressed solely in terms of the basis vectors.

We have

\[
\begin{align*}
  b &= \alpha_1 \hat{c}_1 + \alpha_2 \hat{c}_2 + \alpha_3 \hat{c}_3 + \alpha_4 \hat{c}_4 + \alpha_5 \hat{c}_5 \\
  &= \alpha_1 \hat{c}_1 + \alpha_2 \hat{c}_2 + \alpha_3 (2\hat{c}_1 - 3\hat{c}_2) + \alpha_4 \hat{c}_4 + \alpha_5 (-4\hat{c}_1 + 2\hat{c}_2 + \hat{c}_4) \\
  &= \left(\alpha_1 + 2\alpha_3 - 4\alpha_5\right)\hat{c}_1 + \left(\alpha_2 - 3\alpha_3 + 2\alpha_5\right)\hat{c}_2 + \left(\alpha_4 + \alpha_5\right)\hat{c}_4
\end{align*}
\]

which is of the form

\[
\beta_1 \hat{c}_1 + \beta_2 \hat{c}_2 + \beta_3 \hat{c}_4
\]

a LC of \( \text{(just) } \hat{c}_1, \hat{c}_2 \text{ and } \hat{c}_4 \).
1. continued:

d) What condition(s) on the $b_i$'s are there in order that the system has a solution?

The last row of the RREF matrix given on the previous page will yield an inconsistency UNLESS $33b_1 - 8b_2 + 2b_3 + b_4 = 0$

Is the only condition

in Col($A$)? If so express it as a LC of the basis vectors.

Does it satisfy the condition in (d)? check:

$33 \cdot 3 - 8 \cdot 10 + 2 \cdot 10 + 1 = 0$?

$99 - 80 - 20 + 1 = 0 \text{ ?}$ yes. In this case, the

upper 3 rows of the RREF now tell us that (by choosing the free variables $x_3$ and $x_5$

both 0!!!)

\[
\begin{align*}
X_1 &= 268b_1 - 61b_2 + 19b_3 = 268 \cdot 3 - 61 \cdot 10 + 19 \cdot (-10) = 4 \\
X_2 &= -70b_1 + 16b_2 - 5b_3 = -70 \cdot 3 + 16 \cdot 10 - 5 \cdot (-10) = 0 \\
X_4 &= 13b_1 - 3b_2 + b_3 = 13 \cdot 3 - 3 \cdot 10 + (-10) = -1
\end{align*}
\]

so $\begin{bmatrix} 3 \\ -10 \\ -1 \end{bmatrix} = 4c_1 + 0c_2 - c_4$

f) Find a basis for Nul($A$).

This means putting $[A | \vec{0}]$ in RREF and analyzing the result; we get

$$
\begin{bmatrix}
1 & 0 & 2 & 0 & -4 & | & 0 \\
0 & 1 & -3 & 0 & 2 & | & 0 \\
0 & 0 & 0 & 1 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
$$

; here $\vec{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ where $x_3$ & $x_5$ are free.

a basis is therefore $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

g) What is the rank of $A$?

rank = dim of cd space = 3

h) Find a basis for the row space of $A$.

The rows of the RREF span exactly the same subspace of $\mathbb{R}^5$

as do the rows of $A$ itself [this is NOT true for columns of course!]

and its clear that $\left\{ \begin{bmatrix} 1, 0, 2, 0, -4 \end{bmatrix}, \begin{bmatrix} 0, 1, -3, 0, 2 \end{bmatrix}, \begin{bmatrix} 0, 0, 0, 1, 1 \end{bmatrix} \right\}$ are a basis.

(Note the vectors are written "cyclically")
2. Suppose $T : \mathbb{R}^p \rightarrow \mathbb{R}^q$ is a linear transformation with $A$ from problem 1 as its matrix.

a) What are the values of $p$ and $q$?

\[
\begin{align*}
p &= 5 \\
q &= 4
\end{align*}
\]

b) Find a basis for the kernel of $T$.

The Kernel of $T$ is the set of vectors \( \{ \hat{x} \in \mathbb{R}^5 \mid T(\hat{x}) = \mathbf{0} \} \)

this is the same as \( \text{Null}(A) \), and in problem 1f, a basis of that was shown to be

\[
\begin{pmatrix}
-2 \\
3 \\
1 \\
0 \\
-1
\end{pmatrix}
\]

c) Is \[
\begin{pmatrix}
3 \\
10 \\
-10 \\
1
\end{pmatrix}
\]
in the kernel of $T$? Explain. (Hint: the reason is very very short)

Since the domain $T$ is $\mathbb{R}^5$ and \[
\begin{pmatrix}
3 \\
10 \\
-10 \\
1
\end{pmatrix}
\]

Of course it can't be in the kernel.

c) Is \[
\begin{pmatrix}
3 \\
-10 \\
10 \\
1
\end{pmatrix}
\]
in the range of $T$? Explain. (Watch the sign changes)

We need to know if there is some vector $\hat{x} \in \mathbb{R}^5$ s.t. $T(\hat{x}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix}$

This is the same as asking, "does $A\hat{x} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix}$ have a soln"?

Checking our inconsistency test from problem 1d, we find

\[
33 \cdot 3 - 8(-10) + 2(10) + 1 \cdot 1 = 99 + 80 + 20 + 1 \neq 0 \,
\]

so no, there can't be a soln to $T(\hat{x}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix}$

d) Is $T$ a one-to-one LT? Explain.

it is if and only if its Kernel is just \( \{ \mathbf{0} \} \).

But in 2b we see this is not the case...

e) Is $T$ onto $\mathbb{R}^q$? Explain.

no. in (c) we have an example of a vector which is not in the range of $T$. (namely \[
\begin{pmatrix}
3 \\
-10
\end{pmatrix}
\]

So \[ \neq 0 \] there can't be a soln to $T(\hat{x}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix}$.
3. For each subset $H$ of the vector space $V$ below, if $H$ is a subspace of $V$, find a basis for it, or explain why there is no basis (e.g., maybe the space has infinite dimension); if $H$ is not a subspace of $V$ explain why not.

a) Let $H$ be all polynomials in $V = \mathbb{P}_4$ which have no $x^2$ term.

$$H \text{ is a subspace with basis } \{ x^4, x^3, x, 1 \}$$

($H$ is a 4-dim. subspace of $V$)

b) Let $H$ be all vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} 3a + 2b \\ 5ab \\ 0 \end{bmatrix}$, where $a$ and $b$ are arbitrary.

$H$ is not a subspace... bonus question...

... 

c) Let $H$ be all functions in $F$ which have the positive $x$ axis as a horizontal asymptote.

$H$ is a subspace. (Note the $0$ vector is in $H$)

It is infinite dimensional, however.

\[ \text{\small $\rightarrow$} \quad \text{\small $\leftarrow$} \quad \text{\small $\leftarrow$} \quad \text{\small $\rightarrow$} \]

$\text{\small $\rightarrow$}$ has the $x$-axis as a horizontal asymptote

d) Let $H = \text{Col}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

Since $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $H$ is a 2-dim subspace of $\mathbb{R}^2$, that is, $H$ is all of $\mathbb{R}^2$. A basis is therefore $\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \}$

e) Let $H = \text{Nul}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

$H$ is a subspace; note that $0$ is the only vector in $\text{Nul}(A)$; $\{ 0 \}$ has no basis.
4. Let \( M = \begin{bmatrix} 10 & 0 & 2 & 4 \\ 4 & 0 & 0 & 3 \\ 5 & 2 & 1 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix} \). Find the determinants of each of the following matrices:

a) \( M = -2 \begin{vmatrix} 10 & 2 & 4 \\ 4 & 0 & 3 \\ 3 & 0 & 3 \end{vmatrix} = (-2)(-2) \begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix} = (-2)(-2)(12-9) = 4 \cdot 3 = 12 \)

b) \( MM \) ("\( M \) times \( M \))

Since \( \det(AB) = \det(A) \cdot \det(B) \) we get: \( 144 \)

c) \( M^{-1} \) since \( \det(M^{-1}) = \frac{1}{\det M} \), we get \( 1/12 \)

(try to assume \( \det M \neq 0 \), of course)

d) The transpose of \( M \).

\[ \det M = \det(M^T) \Rightarrow \boxed{12} \text{ is the answer.} \]

e) \( M_2 \) obtained from \( M \) by multiplying every entry of \( M \) by 10.

Every time a row is multiplied by 10, the det increases by a factor of 10.

Since there are four rows, we get \( 12 \times 10^4 = 120,000 \).

f) \( M_3 \) obtained from \( M \) by adding a copy of row 1 to each of the other rows.

\( \det \) doesn't change. So \( \boxed{12} \)

g) \( M_4 \) obtained from \( M \) by pushing each row down by one and moving the last row to the top.

(How many row swaps does this take?)

3 row swaps means the det changes sign 3 times,

Ultimately leaving us with \( \boxed{12} \)
5. True or false quick answers. Write "T" if the statement is always True, or "F" if it is not always true, at the end of each.

a) Every finite dimensional vector space has a basis.
   \[ \text{FALSE} \quad \{ \vec{0} \} \text{ is finite dim. but has no basis.} \quad (\text{dim} \neq 0) \]

b) If \( V \) has a basis with 5 elements and \( S \) is a subset of \( V \) with 6 vectors then \( S \) cannot be linearly independent (LI).
   \[ \text{TRUE} \]

c) If \( V \) has a basis with 5 elements and \( S \) is a subset of \( V \) with 6 vectors then \( S \) must span \( V \).
   \[ \text{FALSE} \quad S \text{ might be } \{ \vec{v}, 2\vec{v}, 3\vec{v}, 4\vec{v}, 5\vec{v}, 6\vec{v} \}, \text{ but the span of } S \text{ would just be a one dim. space.} \]

d) The dimension of \( P_5 \) is 5.
   \[ \text{FALSE} \quad \text{there are 6 vectors in the basis } \{ x^5, x^4, x^3, x^2, x, 1 \} \text{ } \text{six vectors over scalar multipled } \vec{v} \]

e) If \( \text{Dim}(V) = 6 \) and \( S \) is a linearly independent subset of \( V \) with 5 vectors then \( S \) can be "expanded" to a basis of \( V \).
   \[ \text{TRUE} \quad \text{(i.e. you can find another vector } \vec{v} \text{ in } V \text{ s.t. } S \cup \{ \vec{v} \} \text{ is a basis of } V \text{.)} \]

f) If \( \text{Dim}(V) = 6 \) and \( S \) is a subset of \( V \) with 7 vectors then \( S \) can be "whittled down" to a basis of \( V \).
   \[ \text{FALSE} : \text{ you'd need to know that these 7 vectors span } V \text{ first!} \]

g) If \( V \) is infinite dimensional then it has a basis with infinitely many elements.
   \[ \text{FALSE} \]

h) If \( \text{Dim } V = 0 \), then \( \{ \vec{0} \} \) is a basis of \( V \).
   \[ \text{FALSE, although } \{ \vec{0} \} \text{ spans } V, \text{ it is NOT a L.I. set!!! (why not?)} \]