MATH 205A Nov. 9, 2005

NAME: suggested solutions

EXAM 3

1	
2	
3	
4	
5	
TOTAL:	

1. circle your final answer
2. BE NEAT
3. READ the questions
4. show ALL work

GOOD LUCK!

1. Consider the system of equations:

$$\begin{aligned} x_1 + 4x_2 - 10x_3 + x_4 \quad + 5x_5 &= b_1 \\ 5x_1 + 21x_2 - 53x_3 + 10x_4 + 32x_5 &= b_2 \\ 2x_1 + 11x_2 - 29x_3 + 18x_4 + 32x_5 &= b_3 \\ 3x_1 + 14x_2 - 36x_3 + 11x_4 + 27x_5 &= b_4 \end{aligned}$$

Let A be the matrix of coefficients for this system and let $\mathbf{c}_i$ denote the i^{th} column of A. It's true that $[A|I_4]$ is row equivalent to:

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 2 & 0 & -4 & 268 & -61 & 19 & 0 \\ 0 & 1 & -3 & 0 & 2 & -70 & 16 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 & 13 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33 & -8 & 2 & 1 \end{array}\right].$$

a) Which columns of A form a basis for Col(A)? (Answer using $\mathbf{c}_1$, $\mathbf{c}_2$, etc, instead of writing out the columns).

$\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_4$ (this is NOT the only answer!!)

b) Express each column $\mathbf{c}_i$ of A which is *not* in your basis as a linear combination (LC) of the basis vectors. (Again, answer using $\mathbf{c}_1$, $\mathbf{c}_2$, etc, instead of writing out the columns).

$\vec{c}_3 = 2\vec{c}_1 - 3\vec{c}_2$

and $\vec{c}_5 = -4\vec{c}_1 + 2\vec{c}_2 + 1\vec{c}_4$

c) Use part (b) to show how $\mathbf{b} = \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \alpha_3\mathbf{c}_3 + \alpha_4\mathbf{c}_4 + \alpha_5\mathbf{c}_5$, a LC of all five columns, can be expressed solely in terms of the basis vectors.

We have
$$\begin{aligned} \vec{b} &= \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \alpha_3\vec{c}_3 + \alpha_4\vec{c}_4 + \alpha_5\vec{c}_5 \\ &= \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \alpha_3(2\vec{c}_1 - 3\vec{c}_2) + \alpha_4\vec{c}_4 + \alpha_5(-4\vec{c}_1 + 2\vec{c}_2 + \vec{c}_4) \\ &= \underbrace{(\alpha_1 + 2\alpha_3 - 4\alpha_5)}_{\beta_1}\vec{c}_1 + \underbrace{(\alpha_2 - 3\alpha_3 + 2\alpha_5)}_{\beta_2}\vec{c}_2 + \underbrace{(\alpha_4 + \alpha_5)}_{\beta_4}\vec{c}_4 \end{aligned}$$

which is of the form $\beta_1\vec{c}_1 + \beta_2\vec{c}_2 + \beta_4\vec{c}_4$,

a L.C. of (just) $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_4$.

1. continued:

d) What condition(s) on the b_i's are there in order that the system has a solution?

The last row of the RREF matrix given on the previous page will yield an inconsistency UNLESS $\boxed{33b_1 + -8b_2 + 2b_3 + b_4 = 0}$

This is the only condition

e) Is $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix}$ in Col(A)? If so express it as a LC of the basis vectors.

Does it satisfy the condition in (d)? check:

$33 \cdot 3 - 8 \cdot 10 + 2 \cdot (-10) + 1 = 0?$

$99 - 80 - 20 + 1 = 0?$ yes. in this case, the upper 3 rows of the RREF now tell us that (by choosing the free variables x_3 and x_5 both 0 !!!)

$$\left\{\begin{array}{l} x_1 = 268b_1 - 61b_2 + 19b_3 = 268 \cdot 3 - 61 \cdot 10 + 19 \cdot (-10) = 4 \\ x_2 = -70b_1 + 16b_2 - 5b_3 = -70 \cdot 3 + 16 \cdot 10 - 5 \cdot (-10) = 0 \\ x_4 = 13b_1 - 3b_2 + b_3 = 13 \cdot 3 - 3 \cdot 10 + (-10) = -1 \end{array}\right\} \text{ so } \begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix} = 4\vec{c}_1 + 0\vec{c}_2 - \vec{c}_4$$

f) Find a basis for Nul(A).

This means putting $[A|\vec{0}]$ in RREF and analyzing the result; we get

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -4 & 0 \\ 0 & 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]; \text{ here } \vec{x} = x_3\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 4 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } x_3 \,\&\, x_5 \text{ are free.}$$

a basis is therefore $\left\{\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$

g) What is the rank of A? rank = dim of col space = 3

h) Find a basis for the *row* space of A.

The rows of the RREF span exactly the same subspace of $\mathbb{R}^5$ as do the rows of A itself [this is NOT true for columns of course!]

and its clear that $\left\{\begin{array}{l} [1, 0, 2, 0, -4] \\ [0, 1, -3, 0, 2] \\ [0, 0, 0, 1, 1] \end{array}\right\}$ are a basis.

(note the vectors are written "sideways")

2. Suppose $T : \mathbf{R}^p \to \mathbf{R}^q$ is a linear transformation with A from problem 1 as its matrix.

a) What are the values of p and q?

$p = 5 \quad q = 4 \qquad \mathbb{R}^5 \xrightarrow{T} \mathbb{R}^4$

b) Find a basis for the kernel of T.

The Kernel of T is the set of vectors $\{\vec{x} \in \mathbb{R}^5 \mid T(\vec{x}) = \vec{0}\}$
this is the same as $Nul(A)$, and in problem 1f, a basis of that was shown to be $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

c) Is $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix}$ in the kernel of T? Explain. (Hint: the reason is very very short)

Since the domain of T is $\mathbb{R}^5$ and $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix} \notin \mathbb{R}^5$, of course it can't be in the Kernel.

c) Is $\begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$ in the range of T? Explain. (Watch the sign changes)

We need to know if there is some vector $\vec{x} \in \mathbb{R}$ s.t. $T(\vec{x}) = \begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$

(is there a vector here, which T sends to .... $\mathbb{R}^5 \; (\vec{x}) \longrightarrow \left(\begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}\right) \mathbb{R}^4$)

This is the same as asking, "does $A\vec{x} = \begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$ have a sol'n"?
Checking our inconsistency test from problem 1d, we find

$33 \cdot 3 - 8(-10) + 2(10) + 1 \cdot 1$
$= 99 + 80 + 20 + 1 \neq 0$!!!

so NO, there can't be a sol'n to $T(\vec{x}) = \begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$

d) Is T a one-to-one LT? Explain.

it is if and only if its Kernel is just $\{\vec{0}\}$.
But in 2b we see this is not the case...

e) Is T *onto* $\mathbf{R}^q$? Explain.

no. in (c) we have an example of a vector which is not in the range of T. (namely $\begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$

3. For each subset H of the vector space V below, if H is a sub*space* of V, find a basis for it, or explain why there is no basis (eg, maybe the space has infinite dimension); if H is not a sub*space* of V explain why not.

a) Let H be all polynomials in $V = \mathbf{P}_4$ which have no x^2 term.

H <u>is</u> a subspace with basis $\{x^4, x^3, x, 1\}$

(it's a 4 dim. subspace of V)

b) Let H be all vectors in $\mathbf{R}^3$ of the form $\begin{bmatrix} 3a+2b \\ 5ab \\ 0 \end{bmatrix}$, where a and b are arbitrary.

H is <u>not</u> a subspace... bonus question... ...

c) Let H be all functions in $\mathbf{F}$ which have the positive x axis as a horizontal asymptote.

H <u>is</u> a subspace. (note the $\vec{0}$ vector <u>is</u> in H

$\vec{0}$ has the x-axis as a horizontal asymptote yes indeed!)

~~H~~ It is infinite dimensional, however.

d) Let $H = \text{Col}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

Since $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, H is a 2 dim subspace of $\mathbb{R}^2$, that is,
H is <u>all</u> of $\mathbb{R}^2$: a basis is therefore $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$

e) Let $H = \text{Nul}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

H <u>is</u> a subspace; note that $\vec{0}$ is the <u>only</u> vector in $Nul(A)$; $\{\vec{0}\}$ has NO <u>Basis</u>!

4. Let $M = \begin{bmatrix} 10 & 0 & 2 & 4 \\ 4 & 0 & 0 & 3 \\ 5 & 2 & 1 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$. Find the determinants of each of the following matrices:

a) M

$$= -2\begin{vmatrix} 10 & 2 & 4 \\ 4 & 0 & 3 \\ 3 & 0 & 3 \end{vmatrix} = (-2)(-2)\begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix} = (-2)(-2)(12-9) = 4 \cdot 3 = 12$$

b) MM ("M times M")

since $\det(AB) = \det(A) \cdot \det(B)$ we get: 144

c) M^{-1} since $\det(M^{-1}) = \frac{1}{\det M}$, we get $1/12$

(this assumese $\det M \neq 0$, of course)

d) The transpose of M.

$\det M = \det(M^T) \Rightarrow$ 12 is the answer.

e) M_2 obtained from M by multiplying *every* entry of M by 10.

every time a <u>row</u> is multiplied by 10, the <u>det</u> increases by a factor of 10. Since there are four rows, we get $12 \times 10^4 = 120{,}000$.

f) M_3 obtained from M by adding a copy of row 1 to each of the other rows.

<u>det</u> doesn't change. so 12

g) M_4 obtained from M by pushing each row down by one and moving the last row to the top. (How many row swaps does this take?)

3 row swaps means the det changes sign 3 times, ultimately leaving us with -12

5. True or false quick answers. Write "T" if the statement is always True, or "F" if it is not always true, at the end of each.

a) Every finite dimensional vector space has a basis.

FALSE $\{\vec{0}\}$ is finite dim. but has no basis. (dim is 0)

b) If V has a basis with 5 elements and S is a subset of V with 6 vectors then S cannot be linearly independent (LI).

TRUE

c) If V has a basis with 5 elements and S is a subset of V with 6 vectors then S must span V.

FALSE S might be $\underbrace{\{\vec{v}, 2\vec{v}, 3\vec{v}, 4\vec{v}, 5\vec{v}, 6\vec{v}\}}_{\text{six vectors}}$, but the span of S would just be a one dim. space (all scalar multiples of $\vec{v}$)

d) The dimension of $\mathbf{P}_5$ is 5.

FALSE there are 6 vectors in the basis $\{x^5, x^4, x^3, x^2, x, 1\}$

e) If $\mathrm{Dim}(V) = 6$ and S is a *linearly independent* subset of V with 5 vectors then S can be "expanded" to a basis of V.

TRUE (i.e. you can find another vector $\vec{v}$ in V s.t. $S \cup \{\vec{v}\}$ is a basis of V.

f) If $\mathrm{Dim}(V) = 6$ and S is a subset of V with 7 vectors then S can be "whittled down" to a basis of V.

FALSE: you'd need to know that those 7 vectors span V first!

g) If V is infinite dimensional then it has a basis with infinitely many elements.

FALSE

h) If $\mathrm{Dim}\, V$ is 0, then $\{\mathbf{0}\}$ is a basis of V.

FALSE. although $\{\vec{0}\}$ spans V, it is NOT a L.I. set!!! (why not?)