1. Consider the system of equations:

$$\begin{aligned} x_1 + 4x_2 - 10x_3 + x_4 + 5x_5 &= b_1 \\ 5x_1 + 21x_2 - 53x_3 + 10x_4 + 32x_5 &= b_2 \\ 2x_1 + 11x_2 - 29x_3 + 18x_4 + 32x_5 &= b_3 \\ 3x_1 + 14x_2 - 36x_3 + 11x_4 + 27x_5 &= b_4 \end{aligned}$$

Let A be the matrix of coefficients for this system and let $\mathbf{c}_i$ denote the i^{th} column of A. It's true that $[A|I_4]$ is row equivalent to:

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 2 & 0 & -4 & 268 & -61 & 19 & 0 \\ 0 & 1 & -3 & 0 & 2 & -70 & 16 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 & 13 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33 & -8 & 2 & 1 \end{array}\right].$$

a) Which columns of A form a basis for Col(A)? (Answer using $\mathbf{c}_1$, $\mathbf{c}_2$, etc, instead of writing out the columns).

b) Express each column $\mathbf{c}_i$ of A which is *not* in your basis as a linear combination (LC) of the basis vectors. (Again, answer using $\mathbf{c}_1$, $\mathbf{c}_2$, etc, instead of writing out the columns).

c) Use part (b) to show how $\mathbf{b} = \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \alpha_3\mathbf{c}_3 + \alpha_4\mathbf{c}_4 + \alpha_5\mathbf{c}_5$, a LC of all five columns, can be expressed solely in terms of the basis vectors.

1. continued:

d) What condition(s) on the b_i's are there in order that the system has a solution?

e) Is $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix}$ in Col(A)? If so express it as a LC of the basis vectors.

f) Find a basis for Nul(A).

g) What is the rank of A?

h) Find a basis for the *row* space of A.

2. Suppose $T : \mathbf{R}^p \to \mathbf{R}^q$ is a linear transformation with A from problem 1 as its matrix.

a) What are the values of p and q?

b) Find a basis for the kernel of T.

c) Is $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 1 \end{bmatrix}$ in the kernel of T? Explain. (Hint: the reason is very very short)

c) Is $\begin{bmatrix} 3 \\ -10 \\ 10 \\ 1 \end{bmatrix}$ in the range of T? Explain. (Watch the sign changes)

d) Is T a one-to-one LT? Explain.

e) Is T *onto* $\mathbf{R}^q$? Explain.

3. For each subset H of the vector space V below, if H is a sub*space* of V, find a basis for it, or explain why there is no basis (eg, maybe the space has infinite dimension); if H is not a sub*space* of V explain why not.

a) Let H be all polynomials in $V = \mathbf{P}_4$ which have no x^2 term.

b) Let H be all vectors in $\mathbf{R}^3$ of the form $\begin{bmatrix} 3a+2b \\ 5ab \\ 0 \end{bmatrix}$, where a and b are arbitrary.

c) Let H be all functions in $\mathbf{F}$ which have the positive x axis as a horizontal asymptote.

d) Let $H = \text{Col}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

e) Let $H = \text{Nul}(A)$, where $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

4. Let $M = \begin{bmatrix} 10 & 0 & 2 & 4 \\ 4 & 0 & 0 & 3 \\ 5 & 2 & 1 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$. Find the determinants of each of the following matrices:

a) M

b) MM ("M times M")

c) M^{-1}

d) The transpose of M.

e) M_2 obtained from M by multiplying *every* entry of M by 10.

f) M_3 obtained from M by adding a copy of row 1 to each of the other rows.

g) M_4 obtained from M by pushing each row down by one and moving the last row to the top. (How many row swaps does this take?)

5. True or false quick answers. Write "T" if the statement is always True, or "F" if it is not always true, at the end of each.

a) Every finite dimensional vector space has a basis.

b) If V has a basis with 5 elements and S is a subset of V with 6 vectors then S cannot be linearly independent (LI).

c) If V has a basis with 5 elements and S is a subset of V with 6 vectors then S must span V.

d) The dimension of $\mathbf{P}_5$ is 5.

e) If $\mathrm{Dim}(V) = 6$ and S is a *linearly independent* subset of V with 5 vectors then S can be "expanded" to a basis of V.

f) If $\mathrm{Dim}(V) = 6$ and S is a subset of V with 7 vectors then S can be "whittled down" to a basis of V.

g) If V is infinite dimensional then it has a basis with infinitely many elements.

h) If $\mathrm{Dim}\, V$ is 0, then $\{\mathbf{0}\}$ is a basis of V.