MATH 205A • EXAM II

NOVEMBER 7, 2008

PRINT YOUR NAME ↓

suggested solutions

do not write here ↓

1	
2	
3	
4	
5	
TOTAL	

① Show all your work in the spaces provided.

② Be NEAT

③ Clearly mark your final answers

④ READ the questions!

GOOD LUCK!

1. Let $A = \begin{bmatrix} 2 & 5 & b \\ a & 6 & 1 \\ 9 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 2 \\ c & 5 & d \\ 1 & 0 & 9 \end{bmatrix}$ and suppose $AB = \begin{bmatrix} 57 & 25 & 82 \\ 64 & 30 & 33 \\ 49 & 0 & 54 \end{bmatrix}$.

1a. What equations involving one or more of a, b, c, and d need to be solved in order to find a, b, c, and d? Are they all linear?

$$AB = \begin{bmatrix} 10+5c+b & 25 & 4+5d+9b \\ 5a+6c+1 & 30 & 2a+6d+9 \\ 45+4 & 0 & 18+36 \end{bmatrix} \quad \text{so} \quad \begin{cases} 10+5c+b = 57 \\ 5a+6c+1 = 64 \\ 4+5d+9b = 82 \\ 2a+6d+9 = 33 \end{cases}$$ are the eqn's to solve.

yes, these equations are all LINEAR.

1b. Solve the system involving the linear equations in (1a). Use augmented matrices where appropriate; show your augmented matrix/matrices.

the corresponding augmented matrix is:

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 1 & 5 & 0 & 57-10=47 \\ 5 & 0 & 6 & 0 & 64-1=63 \\ 0 & 9 & 0 & 5 & 82-4=78 \\ 2 & 0 & 0 & 6 & 33-9=24 \end{array}\right]$$

the RREF of this is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

telling us that $\begin{cases} a=3 \\ b=7 \\ C=8 \\ d=3 \end{cases}$

2. A model of an economy shows four sectors M, A, T, and H. Output from each sector is distributed as follows: M requires none of H's output, and the other three sectors consume equal amounts of the output of H. None of M's output is used by H and M, A, and T receive equal amounts of M's output. M, T and H use 10, 40 and 30% of A's output respectively; A uses the rest for itself. Finally, T uses none of its own output, half of which is consumed by M and the remainder split between A and H.

2a. What is the exchange table for this model?

M ↓	A ↓	T ↓	H ↓	
1/3	.1	1/2	0	→ M
1/3	.2	1/4	1/3	→ A
1/3	.4	0	1/3	→ T
0	.3	1/4	1/3	→ H

2b. Find a set of equilibrium prices for P_M, P_A, P_T, and P_H so that each sector's income matches its expenses, assuming that P_H is 81.

the equations are

$$\begin{cases} P_M = \frac{1}{3}P_M + \frac{1}{10}P_A + \frac{1}{2}P_T \\ P_A = \frac{1}{3}P_M + \frac{2}{10}P_A + \frac{1}{4}P_T + \frac{1}{3}P_H \\ P_T = \frac{1}{3}P_M + \frac{4}{10}P_A + \frac{1}{3}P_H \\ P_H = \frac{3}{10}P_A + \frac{1}{4}P_T + \frac{1}{3}P_H \end{cases}$$

the corresponding augmented matrix becomes

$$\left[\begin{array}{cccc|c} \frac{1}{3}-1 & \frac{1}{10} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{10}-1 & \frac{1}{4} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{4}{10} & -1 & \frac{1}{3} & 0 \\ 0 & \frac{3}{10} & \frac{1}{4} & \frac{1}{3}-1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} -\frac{2}{3} & \frac{1}{10} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{8}{10} & \frac{1}{4} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{4}{10} & -1 & \frac{1}{3} & 0 \\ 0 & \frac{3}{10} & \frac{1}{4} & -\frac{2}{3} & 0 \end{array}\right]$$

the RREF is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -29/27 & 0 \\ 0 & 1 & 0 & -100/81 & 0 \\ 0 & 0 & 1 & -32/27 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $P_M = 29/27\, P_H$

$P_A = 100/81\, P_H$

$P_T = 32/27\, P_H$

where P_H is free.

if we take $P_H = 81$, we obtain

$$\boxed{\begin{array}{l} P_M = 29/27 \cdot 81 = 87 \\ P_A = 100 \\ P_T = 32/27 \cdot 81 = 96 \end{array}}$$

3. Let $\overline{S}$ be the familiar vector space of all sequences $\mathbf{s} = (s_1, s_2, s_3, s_4, \ldots)$ of real numbers such that all but finitely many of the entries of $\mathbf{s}$ are 0. Suppose $T : \overline{S} \to \overline{S}$ is defined by

$$T(\mathbf{s}) = T((s_1, s_2, s_3, s_4, \ldots)) = (s_1 - s_2, s_2 - s_1, s_3 - s_4, s_4 - s_3, \ldots).$$

3a. Find $T((2, 4, 3, 0, 8, 1, 0, 0, 0, 0, 0, 0 \ldots))$.

$$= (2-4, 4-2, 3-0, 0-3, 8-1, 1-8, 0, 0, 0, 0) = (-2, 2, 3, -3, 7, -7, 0, 0, 0, 0)$$

3b. Show that T is a linear transformation.

part ① Let $\vec{u}$ and $\vec{v}$ be in $\overline{S}$. We need to show $T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v})$. ⊛

Let $\vec{u} = (u_1, u_2, u_3, \ldots)$ and $\vec{v} = (v_1, v_2, v_3, \ldots)$ be in $\overline{S}$.

Now, $T(\vec{u}) + T(\vec{v}) = T((u_1, u_2, u_3, \ldots)) + T((v_1, v_2, v_3, \ldots))$

$= (u_1 - u_2, u_2 - u_1, \ldots) + (v_1 - v_2, v_2 - v_1, \ldots)$

$= (u_1 - u_2 + v_1 - v_2, \ u_2 - u_1 + v_2 - v_1, \ \ldots)$ (1)

and $T(\vec{u}+\vec{v}) = T((u_1, u_2, u_3, \ldots) + (v_1, v_2, v_3, \ldots))$

$= T((u_1 + v_1, u_2 + v_2, \ldots))$

$= (u_1 + v_1 - (u_2 + v_2), \ u_2 + v_2 - (u_1 + v_1), \ \ldots)$

$= (u_1 - u_2 + v_1 - v_2, \ u_2 - u_1 + v_2 - v_1, \ \ldots)$ (2)

Since the expressions in (1) and (2) are identical, ⊛ holds.

part ② Let $\vec{u}$ be as above and let $\alpha \in \mathbb{R}$. We need to show $\alpha T(\vec{u}) = T(\alpha \vec{u})$.

Now, $\alpha T(\vec{u}) = \alpha T((u_1, u_2, u_3, \ldots)) = \alpha (u_1 - u_2, u_2 - u_1, \ldots) =$ (next line)

$= (\alpha(u_1 - u_2), \alpha(u_2 - u_1), \ldots) = (\alpha u_1 - \alpha u_2, \alpha u_2 - \alpha u_1, \ldots) = T((\alpha u_1, \alpha u_2, \ldots))$

$= T(\alpha \vec{u})$ as desired

3c. What form does $\mathbf{s} = (s_1, s_2, s_3, s_4, \ldots)$ have to have in order for $\mathbf{s}$ to be in $\mathrm{Ker}(T)$?

$T(\vec{s}) = \vec{0} \Leftrightarrow (s_1 - s_2, s_2 - s_1, \ldots) = (0, 0, 0, 0, \ldots) \Leftrightarrow s_1 = s_2, \ s_3 = s_4, \ s_5 = s_6$, etc.

3d. What form does $\mathbf{b} = (b_1, b_2, b_3, b_4, \ldots)$ have to have in order for $\mathbf{b}$ to be in $\mathrm{Im}(T)$?

$\vec{b} \in \mathrm{Im}(T) \Leftrightarrow$ there's an $\vec{s}$ s.t. $T(\vec{s}) = \vec{b}$. Since $T(\vec{s}) = (s_1 - s_2, s_2 - s_1, s_3 - s_4, s_4 - s_3, \ldots)$ we need $\begin{cases} s_1 - s_2 = b_1 \\ s_2 - s_1 = b_2 \end{cases}$ i.e. $b_2 = -b_1$, etc. so $\vec{b} = (b_1, -b_1, b_3, -b_3, \ldots)$

3e. Is T one-to-one? *Explain!*

No. for example, $\mathrm{Ker}(T) \neq \{\vec{0}\}$ since $(1, 1, 0, 0, \ldots)$ is also in the Kernel.

3f. Is T onto $\overline{S}$? *Explain!*

No. Not every member of $\overline{S}$ has the special form $(b_1, -b_1, b_3, -b_3, \ldots)$

for example, let $\vec{b} = (1, 2, 3, 4, 0, 0, 0, \ldots)$. There is no $\vec{x} \in \overline{S}$ s.t. $T(\vec{x}) = \vec{b}$.

4. Let $D = \begin{bmatrix} 1 & 1 & 3 & 3 & 8 \\ 1 & 1 & 4 & 5 & 10 \\ 3 & 3 & 11 & 14 & 29 \\ 2 & 2 & 7 & 6 & 16 \end{bmatrix}$ and label its five columns $\mathbf{c}_1, \mathbf{c}_2, \ldots \mathbf{c}_5$.

4a. Find a basis for Col(D). note that RREF(D) is $\begin{bmatrix} 1 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since the pivot columns are #'s 1, 3, & 4, those columns form a basis of Col(D). $\therefore \{\vec{c}_1, \vec{c}_3, \vec{c}_4\}$

4b. Find a basis for Col(RREF(D)).

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4c. Find a basis for Nul(D).

$D\vec{x} = \vec{0} \Leftrightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ when x_2 & x_5 are free.

since these two span Nul(D) and form a L.I. set, they form a basis of Nul(D).

4d. Find a basis for Nul(RREF(D)).

$$\therefore \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Since the solns of RREF(D) $\vec{x} = \vec{0}$ and $D\vec{x} = \vec{0}$ are identical, RREF(D) & D have the same Null Space. $\therefore$ This is a basis for both of them.

4e. What is the rank of D? 3 since a basis for Col(D) has 3 members.

4f. Is $\mathbf{c}_3$ a linear combination of the other columns? Explain. No:

Reason 1: If $\vec{c}_3 = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \alpha_4\vec{c}_4 + \alpha_5\vec{c}_5$, then $\vec{0} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + (-1)\vec{c}_3 + \alpha_4\vec{c}_4 + \alpha_5\vec{c}_5$ shows that a soln of $D\vec{x} = \vec{0}$ exists in which the weight of $\vec{c}_3$ is nonzero. But in 4c we see in any such soln, the weight MUST be 0. so $\vec{c}_3$ can't a L.C. of the other cols.

Reason 2: Row reduction of $[\vec{c}_1\ \vec{c}_2\ \vec{c}_4\ \vec{c}_5 \mid \vec{c}_3]$ shows the underlying system is inconsistent.

5. All questions on this page are "short answer". **** **No** proofs, **no** explanations required! ****

Note that "DNE" (for "Does Not Exist") is a possible answer.

5a. Dim($\mathbf{R}^4$) is?

4

5b. A basis for $\mathbb{P}_3$ is?

$\{1, x, x^2, x^3\}$

5c. Dim(Nul(I_4)) is?

Since Nul$(I_4) = \{\vec{0}_4\}$, and this VS has no basis, we agreed the dim. is 0 (by def'n)

5d. The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is?

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided $ad-bc \neq 0$; otherwise there is no inverse.

5e. A finite set that spans($\overline{S}$) is?

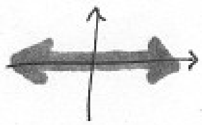
DNE

5f. A finite set that spans $\{\mathbf{0}\}$ is?

$\{\vec{0}\}$ (although this set is not a basis for $\{\vec{0}\}$, it still SPANS $\{\vec{0}\}$. It's just not a LI set)

Let H be the subset of all functions in $\mathbb{F}$ whose graph either never goes below the x-axis.

5g. Is the 0-vector of $\mathbb{F}$ a member of H? (Y/N)

Y (since "$\vec{0}$" = )

5h. Is H closed under vector addition? (Y/N)

Y

5i. Is H closed under scalar multiplication? (Y/N)

N

5j. $\mathbb{P}_3$ is a subspace of $\mathbb{P}_4$ of dimension 4. (T/F)

T

5k. $\mathbf{R}^3$ is a subspace of $\mathbf{R}^4$ of dimension 3. (T/F)

F $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^4$ in the 1st place, dimension notwithstanding!