

Math 105: Review for Exam II - Solutions

1. Find dy/dx for each of the following.

(a) $y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2$$

Note that e^2 , $\ln 2$, and $\arctan 2$ are constants.

(b) $y = \sqrt{x} \cdot \arctan(5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} \arctan(5x) + \frac{5\sqrt{x}}{1+25x^2}$$

(c) $y = \ln(\tan(2^{\cos(x^2)}))$

$$\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})} \cdot \sec^2(2^{\cos(x^2)}) \cdot \ln 2(2^{\cos(x^2)}) \cdot (-\sin(x^2)) \cdot 2x$$

(d) $y = \sin^5\left(\frac{x + e^\pi}{\ln 4 + \arcsin 6x}\right)$

Note that $\sin^5 w = (\sin w)^5$.

$$\frac{dy}{dx} = 5 \sin^4\left(\frac{x + e^\pi}{\ln 4 + \arcsin 6x}\right) \cdot \cos\left(\frac{x + e^\pi}{\ln 4 + \arcsin 6x}\right) \cdot \frac{(1)(\ln 4 + \arcsin 6x) - (x + e^\pi)\left(\frac{1}{\sqrt{1-(6x)^2}} \cdot 6\right)}{(\ln 4 + \arcsin 6x)^2}$$

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) **Find dy/dx .**

Use implicit differentiation.

$$\begin{aligned} 3x^2 + 3y^2 \frac{dy}{dx} &= \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} \left(3y^2 - \frac{9}{2}x\right) &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} &= \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \end{aligned}$$

(b) **Find the equation of the tangent line at the point (1,2).**

We want $y = mx + b$.

$$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$

Now plug in $x = 1$ and $y = 2$ to find b .

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. Evaluate the following limits.

Throughout this solution, the symbol \star will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form $0/0$; this may be $\stackrel{0/0}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or $\stackrel{0/0}{=}$ or "has the form $\frac{0}{0}$, and so, by L'Hopital's Rule, is equal to" or something else. The symbol \heartsuit will serve the same purpose for the indeterminate forms ∞/∞ and $-\infty/\infty$.

(a) $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \star \lim_{x \rightarrow 0} \frac{3 \cos 3x}{5} = \frac{3}{5}$

$$(b) \lim_{x \rightarrow \infty} \frac{e^x}{\ln x} \heartsuit \lim_{x \rightarrow \infty} \frac{e^x}{1/x} = \lim_{x \rightarrow \infty} x e^x = \infty$$

$$(c) \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3x} = \frac{0}{1} = 0$$

Can't use (and don't need) L'Hopital's Rule!

$$(d) \lim_{x \rightarrow 0^+} x^2 \ln x$$

[The 8:00 and 9:30 sections may omit this part.]

This is of the indeterminate form $0 \cdot (-\infty)$ so we rewrite the function as a fraction in order to use L'Hopital's Rule.

$$\lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} \heartsuit \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{x^3}{-2} = \lim_{x \rightarrow 0^+} \frac{x^2}{-2} = 0$$

$$(e) \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{5x^2} \star \lim_{x \rightarrow 0} \frac{4 \sin 4x}{10x} \star \lim_{x \rightarrow 0} \frac{16 \cos 4x}{10} = \frac{16}{10} = \frac{8}{5}$$

4. Suppose the domain of $f(x)$ is all reals and that f has an inverse function $f^{-1}(x)$. Further, suppose that $f(2) = 5$ and $f'(2) = e$. Finally, let $h(x) = 1/f(x)$.

(a) What point must be on the graph of $f^{-1}(x)$?

Since $f(2) = 5$, we also know that $f^{-1}(5) = 2$; therefore, the point $(5, 2)$ is on the graph of $f^{-1}(x)$.

(b) What point must be on the graph of $h(x)$?

Since $f(2) = 5$, we also know that $h(2) = 1/5$; therefore, the point $(2, 1/5)$ is on the graph of $h(x)$.

The moral of the story in parts (a) and (b) is that inverses and reciprocals are not the same.

(c) Give an example of a point that cannot be on the graph of $f(x)$. Do not choose a point with x -value of 2.

Since $f^{-1}(x)$ is a *function*, we know that for each of its inputs (such as 5) there is only one output (in this case, 2). This means that no point of the form $(5, k)$ can be on the graph of $f^{-1}(x)$ (except for $k = 2$). Therefore, no point of the form $(k, 5)$ can be on the graph of $f(x)$ (except for $k = 2$). [Note: this is the same as saying that f must be a one-to-one function; on f , no two x -values correspond to the same y -value.]

(d) What is the value of the derivative of $h(x)$ at $x = 2$?

$$\text{By the Quotient Rule (or Chain Rule if you prefer), } h'(x) = \frac{0 \cdot f(x) - 1 \cdot f'(x)}{[f(x)]^2} = -\frac{f'(x)}{[f(x)]^2}.$$

$$\text{So, } h'(2) = -\frac{f'(2)}{[f(2)]^2} = -\frac{e}{5^2} = -\frac{e}{25}.$$

5. Suppose that $y = f(t)$ is a solution to the differential equation $y' = \frac{1}{\pi} \arcsin t + y^2$ and that

$$f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}. \text{ Find the equation of the tangent line to } f \text{ at } \left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right).$$

We want $y = mx + b$.

$$m = \frac{1}{\pi} \arcsin \frac{\sqrt{2}}{2} + \left(\frac{1}{2}\right)^2 = \frac{1}{\pi} \cdot \frac{\pi}{4} + \frac{1}{4} = \frac{1}{2}, \text{ so } y = \frac{1}{2}x + b.$$

Now plug in $x = \frac{\sqrt{2}}{2}$ and $y = \frac{1}{2}$ to find b .

$$\frac{1}{2} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + b \Rightarrow \frac{1}{2} - \frac{\sqrt{2}}{4} = b$$

Therefore, we have $y = \frac{1}{2}x + \frac{1}{2} - \frac{\sqrt{2}}{4}$.

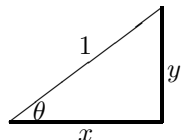
6. Find the following.

- (a) an antiderivative of $y = \frac{5}{\sqrt{1-9x^2}} + x^3 + \cos(2x) + e^3$

$$\frac{5 \arcsin 3x}{3} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^3 x + C$$

- (b) $\tan(\arccos x)$ (simplified as much as possible)

Let $\theta = \arccos x$. That is, θ is the angle whose cosine is x .



$$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1-x^2}$$

$$\tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1-x^2}}{x}$$

7. Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

- (a) Find the x -value(s) of all roots (x -intercepts) of f .

The equation $x^4 e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

- (b) Find the x - and y -value(s) of all critical points and identify each as a local max, local min, or neither.

$$f'(x) = 4x^3 e^x + x^4 e^x$$

$$0 = x^3 e^x (4 + x)$$

$$\Rightarrow x = 0, -4$$

Note that e^x is never 0.

	$x < -4$	$-4 < x < 0$	$4 < x$
f'	positive	negative	positive
f	↗	↘	↗

$$y\text{-values: } f(-4) = 256e^{-4} \approx 4.689, f(0) = 0$$

So, f has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

- (c) Find the x - and y -value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at $(0, 0)$. There is no global maximum because as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Note that as $x \rightarrow -\infty$, $f(x) \rightarrow 0$. You can verify this by using L'Hopital's Rule on x^4/e^{-x} .

- (d) Find the x -value(s) of all inflection points.

$$f''(x) = 12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.}$$

$$0 = e^x (x^4 + 8x^3 + 12x^2)$$

$$0 = e^x x^2 (x^2 + 8x + 12)$$

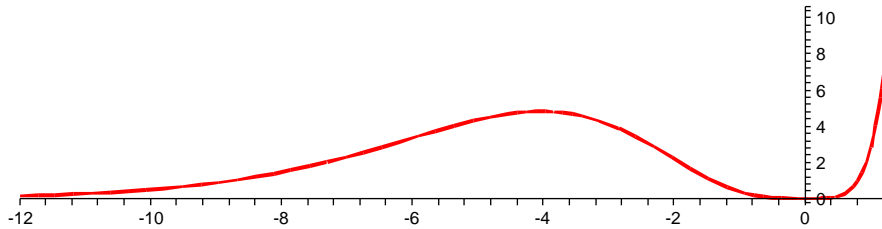
$$0 = e^x x^2 (x+2)(x+6)$$

$$\Rightarrow x = 0, -2, -6$$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
f''	positive	negative	positive	positive
f	concave up	concave down	concave up	concave up

So, the x -values of the inflection points of f are $x = -2$ and $x = -6$ but NOT $x = 0$.

(e) Sketch f .



8. How would your answers to the previous question change if the domain of f were $[-10, 10]$?

There would be a global maximum at $(10, 10^4 e^{10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

9. Use Newton's Method with an initial guess of $x_0 = -1$ to find the next three approximations to a solution of $x^3 + x + 1 = 0$. Then test your final approximation to see if it appears to be close to a root.

$$\text{Recall that } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1}.$$

Using a calculator with initial guess $x_0 = -1$, we get the following.

$$\begin{aligned} x_0 &= -1 \\ x_1 &= -0.75 \\ x_2 &= -0.68604\dots \\ x_3 &= -0.68233\dots \end{aligned}$$

Check: $(-0.68233)^3 + (-0.68233) + 1 \approx 0.0000526$, so our approximation seems to be a good one.

10. Circle *always*, *sometimes*, or *never* to make each statement below correct.

(a) If $f'(1) = 0$ then f *always* / *sometimes* / *never* has a critical point at $x = 1$.

A critical point is where f' is 0 or undefined.

(b) If $f'(2) = 0$ then f *always* / *sometimes* / *never* has a local maximum or local minimum at $x = 2$.

f might instead have a terrace point at $x = 2$; we need f' to change sign at $x = 2$ in order to guarantee a local extremum there.

(c) If $x = 3$ is a critical point of f , then $f'(3)$ is *always* / *sometimes* / *never* 0.

It may also be that $f'(3)$ is undefined.

(d) If $f''(4) = 0$, then f *always* / *sometimes* / *never* has an inflection point at $x = 4$.

We need f'' to change sign at $x = 4$ to guarantee an inflection point there.

For example, if $f(x) = (x-4)^4$ then $f''(4) = 0$ but f has a local minimum rather than an inflection point at $x = 4$.

However, if $f(x) = (x-4)^3$ then $f''(4) = 0$ and f does have an inflection point at $x = 4$.

Also see what happens at $x = 0$ in problem 7(d).

(e) If f has a global maximum at $x = 5$, then $f'(5)$ is *always* / *sometimes* / *never* 0.

$f'(5)$ might also be undefined, or $x = 5$ might be an endpoint of the domain.

- (f) If $f'(6) = 0$ and $f''(6) = -2$, then f always/*sometimes/never* has a local maximum at $x = 6$.

If f is concave down with a horizontal slope at $x = 6$, then f must have a local maximum there.

- (g) If $f'(7) = 0$ and $f''(7) = 0$, then f *always*/sometimes/*never* has a local extremum at $x = 7$.

This means that the second derivative test is inconclusive, so you need to use a different test.

For example, if $f(x) = (x - 7)^4$ then $f'(7) = 0$ and $f''(7) = 0$ and f has a local minimum at $x = 7$.

However, if $f(x) = (x - 7)^3$ then $f'(7) = 0$ and $f''(7) = 0$ and f has an inflection point but not a local extremum at $x = 7$.

11. **The rate of change of a population $P(t)$ of eels is proportional to the size of the population. When the population is 40000, it is growing at a rate of 400 eels per year. At time $t = 0$, the population is 10000.**

- (a) **Write a differential equation whose solution is $P(t)$.**

Rate of change (P') is (=) proportional to (k) size of population (P) means $P' = kP$.

What's the value of k ? When $P = 40000$, we know $P' = 400$. That is, $400 = k \cdot 40000$, so $k = .01$.

Thus, we have $P' = .01P$.

- (b) **Solve your differential equation.**

The general solution is $P(t) = Ae^{.01t}$.

What's the value of A ? When $t = 0$, we know $P = 10000$. That is, $10000 = Ae^0 = A$, so $A = 10000$.

Thus, we have $P(t) = 10000e^{.01t}$.

- (c) **When will the population reach 60000?**

$$60000 = 10000e^{.01t}$$

$$6 = e^{.01t}$$

$$\ln 6 = \ln e^{.01t}$$

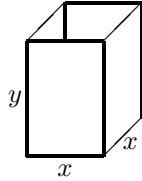
$$\ln 6 = .01t$$

$$t = 100 \ln 6 \approx 179.176 \text{ years}$$

Take \ln of each side.

Recall that $\ln e^z = z$.

12. You are designing an 18 ft^3 box that will have a square bottom and no top. The material for the bottom costs 40 cents per square foot and the material for the sides costs 30 cents per square foot. What dimensions give the least total cost? Be sure to show how you know you have found the minimum.



Goal: minimize cost

Objective function: cost = $C = 40 \cdot x^2 + 30 \cdot 4xy$

We need to get this down to a function of just one variable, so we use the

constraint equation: volume = $18 = x^2y$.

Solving for y , we have $y = \frac{18}{x^2}$.

Substituting this back into the objective function gives $C = 40 \cdot x^2 + 30 \cdot 4x \cdot \frac{18}{x^2} = 40x^2 + \frac{2160}{x}$.

Now that we have C as a function of just one variable, we find its minimum.

$$C'(x) = 80x - \frac{2160}{x^2}$$

$$0 = 80x - \frac{2160}{x^2}$$

$$\frac{2160}{x^2} = 80x$$

$$\frac{2160}{80} = x^3$$

$$3 = x$$

Since C' is negative for $0 < x < 3$ and positive for $3 < x$, we know that the minimum occurs at $x = 3$.

And $y = \frac{18}{x^2} = \frac{18}{3^2} = 2$, so the dimensions are 3 by 3 by 2.