MATH206A MULTIVARIABLE CALCULUS - PROF. P. WONG

EXAM II - NOVEMBER 6, 2007

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	15	
2.	20	
3.	15	
4.	20	
5.	15	
6.	15	
Total	100	

1.(7 pts) (i) Consider the function

$$f(x,y) = \begin{cases} \frac{3xy}{x^4+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether $f(x,y)$ is a continuous at $(0,0)$. Justify your answer. [Hint: try approaching $(0,0)$ from different directions]

Approach $(0,0)$ **along the line** $y = mx$ **for some** $m \neq 0$**. For any** (x,y) **on this line that is different from** $(0,0)$**, we have**

$$f(x,y) = \frac{3x(mx)}{x^4 + m^2x^2} = \frac{3mx^2}{x^2(x^2+m^2)} = \frac{3m}{x^2+m^2}$$

As $(x,y) \to (0,0)$ **along** $y = mx$**,**

$$f(x,y) \to \frac{3m}{m^2} = \frac{3}{m}.$$

Thus for different values of m**,** $f(x,y)$ **would approach to different limits as** $(x,y) \to (0,0)$**. Hence, the limit**

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist and therefore f **is not continuous at** $(0,0)$**.**

(8 pts.)(ii) Consider the vector field $F(x,y,z) = (2xz, -xy, -z)$. Find div F and curl F.

Here, $F = (F_1, F_2, F_3)$ **where** $F_1 = 2xz, F_2 = -xy, F_3 = -z$**. It follows that**

$$\textbf{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= 2z - x - 1.$$

$$\textbf{curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_2 \end{vmatrix}$$
$$= (0-0)\mathbf{i} - (0-2x)\mathbf{j} + (-y-0)\mathbf{k}$$
$$= 2x\mathbf{j} - y\mathbf{k}.$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) = (\cos(\pi x), \sin(\pi x), e^{x+y})$. Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $g(x, y, z) = (x + y, \ln z)$.

(4 pts) (i) Find the Jacobian matrix $J(f)$.

Since $f : \mathbb{R}^2 \to \mathbb{R}^3$, $J(f)$ is a 3×2 matrix.

$$J(f) = \begin{bmatrix} -\pi \sin(\pi x) & 0 \\ \pi \cos(\pi x) & 0 \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

(4 pts) (ii) Find the Jacobian matrix $J(g)$.

Since $g : \mathbb{R}^3 \to \mathbb{R}^2$, $J(g)$ is a 2×3 matrix.

$$J(g) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

(6 pts) Find $J(g \circ f)(0, 1)$.

The chain rule implies that $J(g \circ f)(0, 1) = J(g)(f(0, 1)) \cdot J(f)(0, 1)$. Since $f(0, 1) = (1, 0, e)$, we have

$$J(g \circ f)(0, 1) = J(g)(1, 0, e) \cdot J(f)(0, 1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \frac{1}{e} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \pi & 0 \\ e & e \end{bmatrix} = \begin{bmatrix} \pi & 0 \\ 1 & 1 \end{bmatrix}.$$

(6 pts) Find $J(f \circ g)(1, 0, e)$.

The chain rule implies that $J(f \circ g)(1, 0, e) = J(f)(1, 1) \cdot J(g)(1, 0, e)$. We have

$$J(f \circ g)(1, 0, e) = \begin{bmatrix} 0 & 0 \\ -\pi & 0 \\ e^2 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \frac{1}{e} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\pi & -\pi & 0 \\ e^2 & e^2 & e \end{bmatrix}.$$

3. Let $f(x,y) = x^2e^{-2y}$.

(7 pts) (i) Find the directional derivative $D_{\mathbf{u}}f(1,0)$ of f at the point $(1,0)$ in the direction of $\mathbf{u} = \mathbf{i} + \mathbf{j}$.

The gradient of f **is** $\nabla f = (2xe^{-2y}, -2x^2e^{-2y})$ **so that** $\nabla f(1,0) = (2,-2)$**. It follows that the directional derivative of** f **in the direction of u is**

$$D_{\mathbf{u}}f(1,0) = \nabla f(1,0) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} = (2,-2) \cdot \frac{(1,1)}{\sqrt{2}} = 0.$$

(8 pts) (ii) Find an equation for the line tangent to the level curve $f(x,y) = 1$ at the point $(1,0)$.

The gradient is always perpendicular to the level set so $\nabla f(1,0)$ **is orthogonal to the level curve** $f(x,y) = 1$ **at the point** $(1,0)$**. Thus, the tangent is given by**

$$((x,y) - (1,0)) \cdot \nabla f(1,0) = 0$$

or

$$((x-1), y) \cdot (2,-2) = 0 \quad \textbf{or} \quad y = x - 1.$$

4. Consider the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$.

(4 pts) (i) Find all the critical points of f.

Note that

$$\frac{\partial f}{\partial x} = 3x^2 + 6x = 3x(x+2),$$
$$\frac{\partial f}{\partial y} = 3y^2 - 6y = 3y(y-2).$$

Setting $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (0,0)$ **yields** $(0,0), (0,2), (-2,0), (-2,2)$ **as critical points.**

(8 pts) (ii) For each of the critical point(s) $\mathbf{a}$ found in part (i), find the corresponding Hessian matrix $Hf(\mathbf{a})$.

The Hessian matrix is

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{bmatrix}.$$

Thus, we have

$$Hf(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix} \qquad Hf(0,2) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$Hf(-2,0) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} \qquad Hf(-2,2) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix}.$$

(8 pts) (iii) Use the second derivative test to classify each of the critical point(s) in part (i), i.e., determine whether the critical point is a local max, local min, or saddle point.

The points $(0,0)$ **and** $(-2,2)$ **are both saddle points since the determinant** $\det Hf < 0$ **for each of these points. The point** $(0,2)$ **is a local minimum since** $\det Hf(0,2) = 36 > 0$ **and** $\frac{\partial^2 f}{\partial x^2}(0,2) = 6 > 0$**. The point** $(-2,0)$ **is a local maximum since** $\det Hf(-2,0) = 36 > 0$ **and** $\frac{\partial^2 f}{\partial x^2}(-2,0) = -6 < 0$**.**

5. Let C be a smooth path in $\mathbb{R}^3$ given by the parametrization

$$\mathbf{x}(t) = (2t, t, 2 - 2t)$$

for $0 \le t \le 1$.

(7 pts) (i) Find the length of the path C.

The length of C is given by

$$L(C) = \int_0^1 \|\mathbf{x}'(t)\| dt = \int_0^1 \|(2, 1, -2)\| dt$$
$$= \int_0^1 3 \ dt = 3.$$

(8 pts.) (ii) Suppose a vertical wall is to be built on top of the path C whose height is given by

$$f(x, y, z) = xy + y + z.$$

Find the surface area of this wall.

The function f in terms of the parameter t is given by

$$f(\mathbf{x}'(t)) = (2t)(t) + (t) + (2 - 2t) = 2t^2 - t + 2.$$

Here, $\|\mathbf{x}'(t)\| = 3$.

Thus, the surface area is given by the path integral

$$A = \int_C f(\mathbf{x}'(t)) \|\mathbf{x}'(t)\| dt = \int_0^1 (2t^2 - t + 2) \cdot 3dt$$
$$= 3 \left[\frac{2t^3}{3} - \frac{t^2}{2} + 2t \right]_0^1 = \frac{13}{2}.$$

6. Let C be the arc of the unit circle from $(1,0)$ to $(0,1)$ and $F(x,y) = (x, x^2+y^2)$.

(6 pts.) Write a parametrization $\mathbf{x}(t)$ for the curve C. Be sure to state the range of the parameter.

Consider the parametrization

$$\mathbf{x}(t) = (\cos t, \sin t)$$

where $0 \le t \le \frac{\pi}{2}$**.**

(9 pts.) Find the work done by the force F over the curve C. That is, find $\int_C F(\mathbf{x}) \cdot d\mathbf{x}$.

Note that

$$F(\mathbf{x}(t)) = (\cos t, \cos^2 t + \sin^2 t) \quad \textbf{and} \quad d\mathbf{x} = (-\sin t, \cos t)\ dt.$$

It follows that

$$\begin{aligned}
\int_C F(\mathbf{x}) \cdot d\mathbf{x} &= \int_0^{\frac{\pi}{2}} (\cos t, \cos^2 t + \sin^2 t) \cdot (-\sin t, \cos t)\ dt \\
&= \int_0^{\frac{\pi}{2}} (\cos t, 1) \cdot (-\sin t, \cos t)\ dt \\
&= \int_0^{\frac{\pi}{2}} \cos t \sin t + \cos t\ dt \\
&= \int_0^{\frac{\pi}{2}} \cos t(1-\sin t)\ dt \quad \textbf{(use substitution } w = 1-\sin t\textbf{)} \\
&= -\frac{(1-\sin t)^2}{2}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}.
\end{aligned}$$