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1. Be neat
2. Circle your final answer
3. Read the questions
4. Show all your work

and: Good luck!
1. Find \( \int \frac{x + 4}{(x + 2)(x + 1)^2} \, dx \).

We expect to find constants \( A, B, C \) for which
\[
\frac{x + 4}{(x + 2)(x + 1)^2} = \frac{A}{x + 2} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}
\]
This requires \( \frac{x + 4}{(x + 2)(x + 1)^2} = \frac{A(x + 1)^2 + B(x + 2)(x + 1) + C(x + 2)}{(x + 2)(x + 1)^2} \)

let \( x = -1 \):

\[
3 = A \cdot 0 + B \cdot 0 + C \cdot 1 \Rightarrow C = 3
\]

let \( x = -2 \):

\[
2 = A \cdot 1^2 + B \cdot 0 + C \cdot 0 \Rightarrow A = 2
\]

let \( x = 0 \):

\[
4 = A \cdot 1 + B \cdot 2 + C \cdot 2
\]

\[
4 = 2 \cdot 1 + B \cdot 2 + 3 \cdot 2
\]

\[
4 = 2 + 2B + 6 \Rightarrow B = -2
\]

So
\[
\int \frac{x + 4}{(x + 2)(x + 1)^2} \, dx = \int \frac{2}{x + 2} \, dx + \int \frac{-2}{x + 1} \, dx + \int \frac{3}{(x + 1)^2} \, dx
\]

\[
= 2 \ln |x + 2| - 2 \ln |x + 1| - 3 (x + 1)^{-1} + C
\]

(note: some students thought \((x + 1)^2\) was somehow the indeterminate \(x^2 + 1\), and so set things up incorrectly as \( A \frac{x}{x + 1} + B(x + C) \frac{1}{(x + 1)^2} \))

2. Find \( \int \sqrt{x} \ln x \, dx \).

Try parts: the LIATE "rule" suggests:

Let \( u = \ln x \), so \( du = \frac{1}{x} \, dx \)

Then:

\( dv = \sqrt{x} \, dx \), and \( v = \frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2} \)

The integral becomes
\[
(\ln x) \left( \frac{2}{3} x^{3/2} \right) - \int \frac{2}{3} x^{3/2} \cdot \frac{1}{x} \, dx
\]

\[
= (\ln x) \frac{2}{3} x^{3/2} - \frac{2}{3} \int x^{1/2} \, dx
\]

\[
= (\ln x) \frac{2}{3} x^{3/2} - \frac{2}{3} \frac{2}{3} x^{3/2} + C
\]

\[
= (\ln x) \frac{2}{3} x^{3/2} - \frac{4}{9} x^{3/2} + C
\]
3. Find \( \int \frac{1}{x^2 \sqrt{x^2 - 9}} \, dx \). Hint: It's a trig substitution, probably involving the secant function.

We know \( \sec^2 t - 1 = \tan^2 t \)

so \( 9 \sec^2 t - 9 = 9 \tan^2 t \)

Compare to \( x^2 - 9 \) to get \( x = 3 \sec t \)

Then \( dx = 3 \sec t \tan t \, dt \)

the integral becomes

\[
\int \frac{3 \sec t \tan t \, dt}{9 \sec^2 t \sqrt{9 \sec^2 t - 9}}
\]

became \( 9 \sec^2 t - 9 = 9 \tan^2 t \)

So: \( \sqrt{\text{LHS}} = \sqrt{\text{RHS}} \)

\( = 3 \tan t \)

\[
= \frac{1}{9} \int \frac{1}{\sec t} \, dt \quad \text{(lots of cancellations)}
\]

\[
= \frac{1}{9} \int \cos t \, dt
\]

\[
= \frac{1}{9} \sin t + C
\]

\[
= \frac{1}{9} \sqrt{x^2 - 9} + C
\]

NOTES: the substitution \( x = 3 \sin t \)

CANNOT WORK, because:

1. \( x = 3 \sin t \Rightarrow \)
   - \( x = 9 \sin^2 t \Rightarrow \)
   - \( x^2 - 9 = 9 \sin^2 t - 9 = -9 \cos^2 t \)

2. \( \sqrt{-9 \cos^2 t} \)
   - This is negative
   - \( \sqrt{-9 \cos^2 t} \) makes no sense;
   - It certainly doesn't simplify to "\( -3 \sec t \)"
4A. Let \( f(x) = \sqrt{x} \). Find the second-degree Taylor polynomial \( P_2(x) \) for \( f \) at \( x_0 = 16 \). Show all your work, neatly organized.

\[
\begin{array}{c|c|c|c}
 n & f^{(n)}(x) & f^{(n)}(x_0) & C_n \\
 0 & x^{1/4} & 16^{1/4} = 2 & C_0 = 2 \\
 1 & \frac{1}{4}x^{-3/4} & \frac{1}{4} \cdot 16^{-3/4} = \frac{1}{4} \cdot 2^{-3} = \frac{1}{32} & C_1 = \frac{1}{32} \\
 2 & -\frac{3}{16}x^{-7/4} & -\frac{3}{16} \cdot 16^{-7/4} = -\frac{3}{16} \cdot 2^{-7} = \frac{3}{2048} & C_2 = \frac{-3}{2048} = -\frac{3}{4096} \\
\end{array}
\]

So: \( P_2(x) = 2 + \frac{1}{32} (x-16) - \frac{3}{4096} (x-16)^2 \)

4B. What does Taylor’s theorem give as the maximum possible error committed by \( P_2 \) on the interval \([1,32]\)? (Find your “\( K_3 \)” to two decimal places)

On \([1,32]\), \( f^{(4)}(x) \), i.e. \( \frac{1}{64}x^{-11/4} \), \( < 0.33 \) \( \Rightarrow \) \( K_3 \) is \( \frac{1}{3!} \)

\[
\begin{align*}
\text{so } |P_2(x) - f(x)| &< \frac{K_3}{3!} (x-16)^3 \\
&< \frac{0.33}{3!} (16)^3 \\
&\approx 22.5
\end{align*}
\]

\[\text{This is "maxed out" at } x=32, \text{ when the value is } (32-16)^3 = 16^3\]

4C. What does the actual maximum error appear to be, by graphing?

At about \( x=1 \), the calculator seems to show a difference in the graphs of \( f(x) \) and \( P_2(x) \) \( \approx 0.366 \); this is the largest separation on \([1,32]\).
5. Solve the initial value problem \[ \frac{dy}{dx} = \frac{\sqrt{x \ln x}}{y} \]
\[ y(9) = 4 \]

The differential equation \( \frac{dy}{dx} = \frac{\sqrt{x \ln x}}{y} \) is separable, and we write
\[ y \frac{dy}{dx} = \frac{\sqrt{x \ln x}}{y} \cdot \frac{y}{y} \]
\[ \int y \frac{dy}{dx} \, dx = \int \frac{\sqrt{x \ln x}}{y} \cdot \frac{y}{y} \, dx \]
Thus
\[ y^{\frac{3}{2}} = (\ln x)^{\frac{3}{2}} x^{\frac{3}{2}} - \frac{1}{4} x^{\frac{3}{2}} + C \]
(by problem 2 of this Exam)
\[ y = \pm \sqrt{2(\ln x)^{\frac{3}{2}} x^{\frac{3}{2}} - \frac{1}{4} x^{\frac{3}{2}} + 2C} \]
We can replace 2C by K, but it must stay under the \( \sqrt{ } \)
Since \( y(9) = 4 \) is positive we take the +, we find
\[ 4 = \sqrt{2(\ln 9)^{\frac{3}{2}} \cdot 9^{\frac{3}{2}} - \frac{1}{4} \cdot 9^{\frac{3}{2}} + K} \]
\[ 4 = \sqrt{99.100 - 2.4 + K} \]
\[ 4 = \sqrt{99.100} \]
Finally:
\[ y = \sqrt{\frac{1}{2} (\ln x)^{\frac{3}{2}} x^{\frac{3}{2}} - \frac{1}{4} x^{\frac{3}{2}} - 39.1} \]

6. Determine if the following improper integral converges, and if so, to what. Support your conclusion with an appropriate table.
\[ \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{x^2 - 4}} \, dx \]

The vertical asymptote at \( x = 2 \) tells us this integral is improper and we must instead find:
\[ \lim_{A \to 2^+} \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{x^2 - 4}} \, dx \]
\[ \lim_{A \to 2^+} \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{x^2 - 4}} \, dx = \text{WHAT?} \]

We suspect \( \int \frac{x}{\sqrt{x^2 - 4}} \, dx \) goes to 0."
\[ \lim_{A \to 2^+} \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{x^2 - 4}} \, dx = \text{WHAT?} \]

\[ \frac{1}{2} \int_{\frac{1}{2}}^{1} \frac{2x \, dx}{\sqrt{x^2 - 4}} \]
\[ = \frac{1}{2} \left[ u - \sqrt{u^2 - 4} \right] \]
\[ = \frac{1}{2} \left[ \frac{u}{\sqrt{u^2 - 4}} \right] \]
\[ (\text{ignoring } + C) = \sqrt{x^2 - 4} \]

Since \( \sqrt{12} = 3.4641 \ldots \)

We claim the table shows...
the integral CONVERGES TO \( \sqrt{12} \)

**NOTE:** if you do the following:

While keeping the limits:
\[ \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{x^2 - 4}} \, dx = \ldots \]

Let \( u = x^2 - 4 \):
\[ du = 2x \, dx \]
\[ \int \frac{2x \, dx}{\sqrt{x^2 - 4}} \]
\[ = \frac{1}{2} \int u^{-\frac{1}{2}} \, du \]
\[ = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \frac{u^{\frac{1}{2}}}{2} \]

\[ (\text{ignoring } + C) = \sqrt{x^2 - 4} \]

\[ = \sqrt{12} - \sqrt{A^2 - 4} \]

\[ \text{as } A \to 2^+ \text{ on the left} \]
7. Use an appropriate comparison to determine whether the following integral converges; explain your conclusion in terms of "p-test". \[ \int_{10}^{\infty} \frac{1}{\sqrt{x} - 16} \, dx. \]

We have \( \sqrt{x} - 16 < \sqrt{x} \) for \( x > 16 \) million.

So \( \frac{1}{\sqrt{x} - 16} > \frac{1}{\sqrt{x}} \)

So \( \int_{10}^{\infty} \frac{1}{\sqrt{x} - 16} \, dx \) diverges, if \( \int_{10}^{\infty} \frac{1}{\sqrt{x}} \, dx \) does. But the "p-test" says, since \( p = \frac{1}{2} < 1 \), that \( \int_{10}^{\infty} \frac{1}{\sqrt{x}} \, dx \) does diverge.

\[ \text{So } \int_{10}^{\infty} \frac{1}{\sqrt{x} - 16} \, dx \text{ must also.} \]

**NOTE:** 1. we do not write the meaningless expression \( \int_{10}^{\infty} \frac{dx}{\sqrt{x} - 16} \geq \int_{10}^{\infty} \frac{dx}{\sqrt{x}} \)

2. no table was necessary to show \( \int_{10}^{\infty} \frac{dx}{\sqrt{x}} \) diverges; that's what you KNOW from the "p-test"!

8. Find \( \int \frac{2x^3 + 12x^2 + 22x + 7}{x^2 + 6x + 10} \, dx \).

Since \( \text{deg(numerator)} > \text{deg(denominator)} \), we carry out the indicated division:

\[ \frac{x^2 + 6x + 10}{x^2 + 6x + 10} \]

So the integral becomes:

\[ \int 2x + 7 \, dx \]

\[ \int \frac{2x + 7}{x^2 + 6x + 10} \, dx \]

So the integral becomes:

\[ \int 2x + \frac{7}{x^2 + 6x + 10} \, dx \]

\[ X^2 + \int \frac{2x + 6}{x^2 + 6x + 10} \, dx + \int \frac{7}{x^2 + 6x + 10} \, dx + C \]

let \( u = x^2 + 6x + 10 \)

\( du = 2x + 6 \, dx \)

in the form

\[ \int \frac{du}{u} = \ln|u| = \ln|x^2 + 6x + 10| + C \]

**Final answer:**

\[ X^2 + \ln|x^2 + 6x + 10| + \text{arctan}(x + 3) + C \]